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Poisson and Hamiltonian superpairs over polarized associative algebras

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Abstract

A Poisson superpair is a pair of Poisson superalgebra structures on a supercommutative associative algebra, whose any linear combination is also a Poisson superalgebra structure. In this paper, we first construct certain linear and quadratic Poisson superpairs over a semi-finitely-filtered polarized $\mathbb{Z}_2$-graded associative algebra. Then we give a construction of certain Hamiltonian superpairs in the formal variational calculus over any finite-dimensional $\mathbb{Z}_2$-graded associative algebra with a supersymmetric nondegenerate associative bilinear form. Our constructions are based on the Adler mapping in a general sense. Our results in this paper can be viewed as noncommutative generalizations of the Adler–Gel'fand–Dikii Hamiltonian pair.

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1. Introduction

In the theory of completely integrable systems, one of the most beautiful structures is the Adler–Gel'fand–Dikii Hamiltonian pair, which was constructed through the Adler mapping [1, 13]. Such a Hamiltonian pair gives a pair of Poisson structures on the quotient space of the differential polynomial algebra of scalar differential operators with fixed order modulo its subspace of total differential polynomials (flux). They do not form Poisson algebra structures because the quotient space does not form an associative algebra.

Gel'fand and Dorfman [15] generalized the Adler–Gel'fand–Dikii Hamiltonian pair to that over differential operators with fixed order and the coefficients in a matrix algebra. The first Hamiltonian structure for the Kadomtsev–Petviashvili hierarchy was suggested by Watanabe [19]. Dickey [6] added the second Hamiltonian structure for the Kadomtsev–Petviashvili hierarchy. These two structures form an analogue of the Adler–Gel'fand–Dikii Hamiltonian pair over scalar pseudo-differential operators of positive order one and infinite negative order. Radul [18] generalized it over scalar pseudo-differential operators of finite positive order and infinite negative order.

Manin and Radul [17] gave a supersymmetric extension of the Kadomtsev–Petviashvili hierarchy. Das and Huang [3] essentially partially generalized the Adler–Gel'fand–Dikii's construction over scalar differential operators with fixed order to that over scalar pseudo-differential operators with fixed positive and negative orders. Dofman and Fokas [9] generalized the Adler–Gel'fand–Dikii Hamiltonian pair to that over differential operators with fixed order and the coefficients in an algebra of pseudo-differential operators, in which the algebra plays the same role as a number field. It seems that there is a problem of how to interpret Dorfman and Fokas' results in [9] in terms of the theory of Hamiltonian operators over a field.

In [20], we generalized the theory of Hamiltonian operators to that of Hamiltonian superoperators over fermionic fields. Moreover, we established an analogous theory of supervariables in [21]. In [22], we proved that conformal superalgebras are equivalent to certain linear Hamiltonian superoperator of superfunctions in one real variable.

We observe that the Adler–Gel'fand–Dikii construction was essentially based on a polarization of the algebra of pseudo-differential operators. Their construction could be generalized and applied to more general polarized associative algebras. The first objective of this paper is to construct linear and quadratic Poisson superpairs on supersymmetric polynomial functions of a semi-finitely-filtered polarized $\mathbb{Z}_2$-graded associative algebra. The second objective is to construct certain Hamiltonian superpairs in the formal variational calculus over any finite-dimensional $\mathbb{Z}_2$-graded associative algebra with a supersymmetric nondegenerate associative bilinear form. Our constructions are based on the Adler mapping in a general sense. The results in this paper reveal that there is a deep algebraic essence behind the Adler–Gel'fand–Dikii Hamiltonian pair. In fact, the linear structure in our general analogues of the Adler–Gel'fand–Dikii Hamiltonian pair depends on a central element.

We shall give a more technical introduction in section 2. In section 3, we shall present certain structural properties and constructions of $\mathbb{Z}_2$-graded polarized associative algebras. The Poisson superpairs will be given in section 4. Section 5 is devoted to the Hamiltonian superpairs in the formal variational calculus.

2. Technical background

This section serves as a technical introduction to the whole paper.

Throughout this paper, we let $\mathbb{F}$ be a field with characteristic not equal to two unless it is specified. All the vector spaces (algebras) are assumed over $\mathbb{F}$. Denote by $\mathbb{Z}$ the set of integers and by $\mathbb{N}$ the set of nonnegative integers. For any two integers m_1, m_2, we shall often use the following notation of index throughout this paper:

$$\overline{m_1, m_2} = \begin{cases} \{m_1, m_1+1, m_1+2, ..., m_2\} & \text{if } m_1 \leqslant m_2 \\ \emptyset & \text{if } m_1 > m_2. \end{cases} \tag{2.1}$$

First we introduce the definition of abstract Hamiltonian superoperators. Let $(\mathcal{G}, [\cdot,\cdot])$ be a Lie superalgebra and let M be a $\mathcal{G}$-module. For a positive integer q, a *q-form of $\mathcal{G}$ with values in M* is a multi-linear map $\omega : \mathcal{G}^q = \mathcal{G}\times\cdots\times\mathcal{G} \to M$ for which

$$\omega(\xi_1, \xi_2, \dots, \xi_q) = -(-1)^{ij}\omega(\xi_1, \dots, \xi_{\ell-1}, \xi_{\ell+1}, \xi_\ell, \xi_{\ell+2}, \dots, \xi_q) \tag{2.2}$$

for $\xi_k \in \mathcal{G}$ with $k \in \overline{1 \cdot q} \setminus \{\ell, \ell+1\}$, $\xi_\ell \in \mathcal{G}_i$ and $\xi_{\ell+1} \in \mathcal{G}_j$. We denote by $c^q(\mathcal{G}, M)$ the set of q-forms. Moreover, we define a differential $d\colon c^q(\mathcal{G}, M) \to c^{q+1}(\mathcal{G}, M)$ by

$$\begin{aligned} d\omega(\xi_1, \xi_2, \dots, \xi_{q+1}) = {} & \sum_{\ell=1}^{q+1} (-1)^{\ell+1+i_\ell(i_1+\cdots+i_{\ell-1})}\xi_\ell\omega(\xi_1, ..., \check{\xi}_\ell, ..., \xi_{q+1}) \\ & + \sum_{\ell_1<\ell_2} (-1)^{\ell_1+\ell_2+(i_{\ell_1}+i_{\ell_2})(i_1+\cdots+i_{\ell_1-1})+i_{\ell_2}(i_{\ell_1+1}+\cdots+i_{\ell_2-1})} \\ & \times \omega([\xi_{\ell_1}, \xi_{\ell_2}], \xi_1, ..., \check{\xi}_{\ell_1}, \dots, \check{\xi}_{\ell_2}, \dots, \xi_{q+1}) \end{aligned} \tag{2.3}$$

for $\omega \in c^q(\mathcal{G}, M)$ and $\xi_k \in \mathcal{G}_{i_k}$ with $k \in \overline{1, q+1}$, where the circumflex accent means deleting the term under it. A q-form ω is called *closed* if $d\omega = 0$.

For any $u \in M$, we define a one-form du by

$$du(\xi) = \xi(u) \qquad \text{for } \xi \in \mathcal{G}. \tag{2.4}$$

Let Ω be a subspace of $c^1(\mathcal{G}, M)$ such that $dM \subset \Omega$. Suppose that $H : \Omega \to \mathcal{G}$ is a linear map. We call H $\mathbb{Z}_2$-*graded* if

$$H(\Omega) = H(\Omega)_0 \oplus H(\Omega)_1 \qquad \text{where} \qquad H(\Omega)_i = H(\Omega) \bigcap \mathcal{G}_i. \tag{2.5}$$

Moreover, H is called *super-skew-symmetric* if

$$\phi_1(H\phi_2) = -(-1)^{i_1 i_2}\phi_2(H\phi_1) \qquad \text{where} \qquad H\phi_j \in H(\Omega)_{i_j}. \tag{2.6}$$

For a $\mathbb{Z}_2$-graded super-skew-symmetric linear map $H : \Omega \to \mathcal{G}$, we define a 2-form ω_H on $H(\Omega)$ by

$$\omega_H(H\phi_1, H\phi_2) = \phi_2(H\phi_1) \qquad \text{for } \phi_1, \phi_2 \in \Omega. \tag{2.7}$$

We say that a super-skew-symmetric $\mathbb{Z}_2$-graded linear map $H : \Omega \to \mathcal{G}$ is a *Hamiltonian superoperator* if

(a) the subspace $H(\Omega)$ of $\mathcal{G}$ forms a subalgebra;
(b) the left and right radicals of the form ω_H are $\mathbb{Z}_2$-graded and $d\omega_H \equiv 0$ on $H(\Omega)$.

Two $\mathbb{Z}_2$-graded linear maps $H_1, H_2 : \Omega \to \mathcal{G}$ are called a *Hamiltonian pair* if $\lambda_1 H_1 + \lambda_2 H_2$ is a Hamiltonian superoperator for any $\lambda_1, \lambda_2 \in \mathbb{F}$.

Next we introduce the Adler–Gel'fand–Dikii Hamiltonian pair and the known generalizations. We assume that $\mathbb{F}$ is a field of real numbers or a field of complex numbers. Let k be a positive integer and let $\{u_0, u_1, \ldots, u_{k-1}\}$ be k C^∞-functions in the real variable x. Set

$$u_j^{(m)} = \frac{\mathrm{d}^m u_j}{\mathrm{d}x^m} \qquad \text{for} \quad m \in \mathbb{N}, \ j \in \overline{0, k-1}. \tag{2.8}$$

Denote

$$\mathcal{P} = \mathbb{F}\big[u_j^{(m)} \mid m \in \mathbb{N}, \ j \in \overline{0, k-1}\}\big] \tag{2.9}$$

the differential polynomial algebras of $\{u_j(x) \mid j \in \overline{0, k-1}\}$. We view

$$\frac{\mathrm{d}}{\mathrm{d}x} = \sum_{j \in \overline{0,k-1},\ m \in \mathbb{N}} u_j^{(m+1)} \partial_{u_j^{(m)}} \tag{2.10}$$

as a derivation of $\mathcal{P}$. For convenience, we denote

$$\partial = \frac{\mathrm{d}}{\mathrm{d}x} \tag{2.11}$$

and define the algebra of pseudo-differential operators

$$\mathcal{D} = \left\{ \sum_{l=-\infty}^{n} f_l \partial^l | n \in \mathbb{Z}, \ f_l \in \mathcal{P} \right\} \tag{2.12}$$

with the multiplication determined by

$$(f\partial^m)(g\partial^n) = \sum_{p=0}^{\infty} \binom{m}{p} f g^{(p)} \partial^{m+n-p} \qquad \text{for } f, g \in \mathcal{P}, \quad m, n \in \mathbb{N} \tag{2.13}$$

where

$$g^{(p)} = \partial^p(g). \tag{2.14}$$

Set

$$\mathcal{G} = \sum_{j=0}^{k-1} \mathcal{P}\partial^j \subset \mathcal{D} \tag{2.15}$$

and

$$\partial_X = \sum_{j\in\overline{0,k-1},\ m\in\mathbb{N}} a_j^{(m)} \partial_{u_j^{(m)}} \qquad \text{for} \quad X = \sum_{j=0}^{k-1} a_j\partial^j \in \mathcal{G}. \tag{2.16}$$

As derivations of $\mathcal{P}$,

$$[\partial_X, \partial] = 0 \qquad \text{for} \ \ X \in \mathcal{G}. \tag{2.17}$$

Define

$$\partial_X(Y) = \sum_{j=0}^{k-1} \partial_X(b_j)\partial^j \qquad \text{for} \ \ X, Y = \sum_{j=0}^{k-1} b_j\partial^j \in \mathcal{G}. \tag{2.18}$$

The Lie bracket on $\mathcal{G}$ is defined by

$$[X, Y]_0 = \partial_X(Y) - \partial_Y(X) \qquad \text{for} \ \ X, Y \in \mathcal{G}. \tag{2.19}$$

Set

$$\tilde{\mathcal{P}} = \mathcal{P}/\partial(\mathcal{P}) \tag{2.20}$$

and use the notation

$$\tilde{f} = f + \partial(\mathcal{P}) \qquad \text{for} \ \ f \in \mathcal{P}. \tag{2.21}$$

Moreover, by (2.17), we define an action of $\mathcal{G}$ on $\tilde{\mathcal{P}}$:

$$X(\tilde{f}) = (\partial_X(f))^{\sim} \qquad \text{for} \ \ X \in \mathcal{G},\ \ f \in \mathcal{A}. \tag{2.22}$$

Then $\tilde{\mathcal{P}}$ forms a $\mathcal{G}$-module.

Define

$$\Omega = \sum_{j=0}^{k-1} \partial^{-1-j}\mathcal{P} \subset \mathcal{D} \tag{2.23}$$

and identify it with a subspace of one-forms by

$$\xi(X) = \sum_{j=0}^{k-1} (a_i\alpha_i)^{\sim} \qquad \text{for} \ \ \xi = \sum_{j=0}^{k-1} \partial^{-1-j}\alpha_j \in \Omega,\ \ X = \sum_{j=0}^{k-1} a_j\partial^j \in \mathcal{G}. \tag{2.24}$$

Moreover, we define the projection from $\mathcal{D}$ to $\mathcal{G}$ by

$$\left(\sum_{l=-\infty}^{n} f_l\partial^l\right)_+ = \sum_{l=0}^{n} f_l\partial^l \qquad \text{for} \quad \sum_{l=-\infty}^{n} f_l\partial^l \in \mathcal{D} \ \text{ with } \ n \in \mathbb{N}. \tag{2.25}$$

Set

$$L = \partial^k + \sum_{j=0}^{k-1} u_j\partial^j. \tag{2.26}$$

We define two linear maps $H_1, H_2 : \Omega \to \mathcal{G}$ by

$$H_1(\xi) = ([L, \xi])_+ \qquad H_2(\xi) = (L\xi)_+ L - L(\xi L)_+. \tag{2.27}$$

Then H_1 and H_2 form a Hamiltonian pair, which is called the *Adler–Gel'fand–Dikii Hamiltonian pair* (cf. [1, 7, 13]). The map H_2 is called the *Adler mapping* (cf. [1]).

Gel'fand and Dorfman [15] generalized the Adler–Gel'fand–Dikii Hamiltonian pair over the differential operator L with u_j taking values in an $m \times m$ matrix algebra. Watanabe [19] and Dickey [6] obtained Adler–Gel'fand–Dikii Hamiltonian pairs over the pseudo-differential operator

$$L = \partial + \sum_{j=-\infty}^{0} u_j \partial^j. \tag{2.28}$$

Radul [18] further generalized it over the pseudo-differential operator

$$L = \partial^k + \sum_{j=-\infty}^{k} u_j \partial^j \tag{2.29}$$

for any positive integer k. Das and Huang [3] generalized the Gel'fand–Dikii construction of Hamiltonian pairs essentially by extending $\mathcal{P}$ to the algebra of differential polynomials of $2k$ C^∞-functions $\{u_{-k}, \ldots, u_{-1}, u_0, \ldots, u_{k-1}\}$, taking

$$L = \partial^k + \sum_{j=-k}^{k-1} u_j \partial^j \tag{2.30}$$

and keeping $\mathcal{G}$ and Ω the same as in (2.15) and (2.23) with the new $\mathcal{P}$. Dorfman and Fokas obtained the Adler–Gel'fand–Dikii Hamiltonian pair over the differential operator L with u_j taking values in an algebra of pseudo-differential operators, in which the algebra plays the same role as a number field. It seems that there is a problem of how to interpret Dorfman and Fokas' results in [9] in terms of the theory of Hamiltonian operators over a field.

Let $\mathbb{F}$ be a general field. A $\mathbb{Z}_2$-graded associative algebra

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \tag{2.31}$$

is called a *polarized* $\mathbb{Z}_2$*-graded associative algebra* if $\mathcal{A}$ has a nondegenerate $\mathbb{Z}_2$-graded supersymmetric associative bilinear form $\langle\cdot,\cdot\rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{F}$, that is,

$$\langle \mathcal{A}_0, \mathcal{A}_1 \rangle = \{0\} \qquad \langle u, v \rangle = (-1)^{i_1 i_2} \langle v, u \rangle \qquad \langle uv, w \rangle = \langle u, vw \rangle \tag{2.32}$$

for $u \in \mathcal{A}_{i_1},\ v \in \mathcal{A}_{i_2},\ w \in \mathcal{A}$, and contains two $\mathbb{Z}_2$-graded isotropic subalgebras $\mathcal{A}^+$ and $\mathcal{A}^-$, namely,

$$\langle \mathcal{A}^+, \mathcal{A}^+ \rangle = \{0\} \qquad \langle \mathcal{A}^-, \mathcal{A}^- \rangle = \{0\} \tag{2.33}$$

such that

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-. \tag{2.34}$$

Expressions (2.33) and (2.34) are called a *polarization of* $\mathcal{A}$.

An associative algebra $\mathcal{B}$ is called *supercommutative* if $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ is a $\mathbb{Z}_2$-graded algebra such that

$$uv = (-1)^{i_1 i_2} vu \qquad \text{for } u \in \mathcal{B}_{i_1},\ v \in \mathcal{B}_{i_2}. \tag{2.35}$$

A *Poisson superalgebra* is a supercommutative associative $\mathcal{B}$ with another algebraic operation $\{\cdot,\cdot\}$, called a *Poisson superbracket*, such that $(\mathcal{B}, \{\cdot,\cdot\})$ forms a Lie superalgebra and the following compatibility condition holds

$$\{u, vw\} = \{u, v\}w + (-1)^{i_2 i_2} v\{u, w\} \qquad \text{for } u \in \mathcal{B}_{i_1},\ v \in \mathcal{B}_{i_2},\ w \in \mathcal{B}. \tag{2.36}$$

The main purpose of this paper is to give a more systematic study in generalizations of the Adler–Gel'fand–Dikii construction of Hamiltonian pairs over a polarized $\mathbb{Z}_2$-graded associative algebra. In particular, we obtain pairs of a linear and a quadratic Poisson superalgebra structures on supersymmetric polynomial functions of a semi-finitely-filtered polarized $\mathbb{Z}_2$-graded associative algebras.

3. Polarized associative algebras

In this section, we shall present certain structural properties and constructions of $\mathbb{Z}_2$-graded polarized associative algebras.

As we shall show below, the structure of a $\mathbb{Z}_2$-graded polarized associative algebra is determined by a certain compatible pair of $\mathbb{Z}_2$-graded associative algebra structures on a vector space.

Let

$$\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^- \tag{3.1}$$

be a polarized $\mathbb{Z}_2$-graded associative algebra with the nondegenerate $\mathbb{Z}_2$-graded supersymmetric associative bilinear form $\langle \cdot, \cdot \rangle$. Set

$$\mathcal{A}_i^{\pm} = \mathcal{A}_i \bigcap \mathcal{A}^{\pm} \qquad \text{for } i \in \mathbb{Z}_2. \tag{3.2}$$

Take a basis $\{\varsigma_{i,j}^+ | j \in I_i\}$ of $\mathcal{A}_i^+$ for $i \in \mathbb{Z}_2$, where I_i are index sets. Suppose that $\mathcal{A}_i^-$ has a dual basis $\{\varsigma_{i,j}^- \mid j \in I_i\}$ with respect to the basis $\{\varsigma_{i,j}^+ \mid j \in I_i\}$ of $\mathcal{A}_i^+$ for $i \in \mathbb{Z}_2$, that is,

$$\langle \varsigma_{i_1,j_1}^-, \varsigma_{i_2,j_2}^+ \rangle = \delta_{i_1,i_2}\delta_{j_1,j_2} \qquad \text{for } i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in I_{i_1},\ j_2 \in I_{i_2}. \tag{3.3}$$

This assumption trivially holds when $\mathcal{A}$ is finite-dimensional.

Write

$$\varsigma_{i_1,j_1}^{\pm}\varsigma_{i_2,j_2}^{\pm} = \sum_{j_3 \in I_{i_1+i_2}} a_{i_1,j_1;i_2,j_2}^{\pm,j_3}\varsigma_{i_1+i_2,j_3}^{\pm} \qquad \text{for } i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in I_{i_1},\ j_2 \in I_{i_2}. \tag{3.4}$$

Then

$$\langle \varsigma_{i_1,j_1}^+\varsigma_{i_2,j_2}^-, \varsigma_{i_1+i_2,j_3}^- \rangle = \langle \varsigma_{i_1,j_1}^+, \varsigma_{i_2,j_2}^-\varsigma_{i_1+i_2,j_3}^- \rangle = (-1)^{i_1} a_{i_2,j_2;i_1+i_2,j_3}^{-,j_1} \tag{3.5}$$

$$\langle \varsigma_{i_1+i_2,j_3}^+, \varsigma_{i_1,j_1}^+\varsigma_{i_2,j_2}^- \rangle = \langle \varsigma_{i_1+i_2,j_3}^+\varsigma_{i_1,j_1}^+, \varsigma_{i_2,j_2}^- \rangle = (-1)^{i_2} a_{i_1+i_2,j_3;i_1,j_1}^{+,j_2} \tag{3.6}$$

$$\langle \varsigma_{i_1,j_1}^-\varsigma_{i_2,j_2}^+, \varsigma_{i_1+i_2,j_3}^+ \rangle = \langle \varsigma_{i_1,j_1}^-, \varsigma_{i_2,j_2}^+\varsigma_{i_1+i_2,j_3}^+ \rangle = a_{i_2,j_2;i_1+i_2,j_3}^{+,j_1} \tag{3.7}$$

$$\langle \varsigma_{i_1+i_2,j_3}^-, \varsigma_{i_1,j_1}^-\varsigma_{i_2,j_2}^+ \rangle = \langle \varsigma_{i_1+i_2,j_3}^-\varsigma_{i_1,j_1}^-, \varsigma_{i_2,j_2}^+ \rangle = a_{i_1+i_2,j_3;i_1,j_1}^{-,j_2} \tag{3.8}$$

for $i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in I_{i_1},\ j_2 \in I_{i_2}$ and $j_3 \in I_{i_1+i_2}$ by (2.32) and (3.3). Thus we have

$$\varsigma_{i_1,j_1}^+\varsigma_{i_2,j_2}^- = \sum_{j_3 \in I_{i_1+i_2}} \left((-1)^{i_2} a_{i_2,j_2;i_1+i_2,j_3}^{-,j_1}\varsigma_{i_1+i_2,j_3}^+ + (-1)^{i_1} a_{i_1+i_2,j_3;i_1,j_1}^{+,j_2}\varsigma_{i_1+i_2,j_3}^- \right) \tag{3.9}$$

$$\varsigma_{i_1,j_1}^-\varsigma_{i_2,j_2}^+ = \sum_{j_3 \in I_{i_1+i_2}} \left(a_{i_2,j_2;i_1+i_2,j_3}^{+,j_1}\varsigma_{i_1+i_2,j_3}^- + a_{i_1+i_2,j_3;i_1,j_1}^{-,j_2}\varsigma_{i_1+i_2,j_3}^+ \right) \tag{3.10}$$

for $i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in I_{i_1}$ and $j_2 \in I_{i_2}$.

Let $i_p \in \mathbb{Z}_2,\ j_p \in I_{i_p}$ with $p = 1, 2, 3$. We have:

$$
\begin{aligned}
(\varsigma^+_{i_1,j_1}\varsigma^+_{i_2,j_2})\varsigma^-_{i_3,j_3} &= \sum_{j_4\in I_{i_1+i_2}} a^{+,j_4}_{i_1,j_1;i_2,j_2}\varsigma^+_{i_1+i_2,j_4}\varsigma^-_{i_3,j_3}\\
&= \sum_{j_4\in I_{i_1+i_2},\ j_5\in I_{i_1+i_2+i_3}} a^{+,j_4}_{i_1,j_1;i_2,j_2}\Big((-1)^{i_3}a^{-,j_4}_{i_3,j_3;i_1+i_2+i_3,j_5}\varsigma^+_{i_1+i_2+i_3,j_5}\\
&\quad + (-1)^{i_1+i_2}a^{+,j_3}_{i_1+i_2+i_3,j_5;i_1+i_2,j_4}\varsigma^-_{i_1+i_2+i_3,j_5}\Big)
\end{aligned}
\tag{3.11}
$$

$$
\begin{aligned}
&\varsigma^+_{i_1,j_1}(\varsigma^+_{i_2,j_2}\varsigma^-_{i_3,j_3})\\
&= \sum_{j_4\in I_{i_2+i_3}} \Big((-1)^{i_3}a^{-,j_2}_{i_3,j_3;i_2+i_3,j_4}\varsigma^+_{i_1,j_1}\varsigma^+_{i_2+i_3,j_4} + (-1)^{i_2}a^{+,j_3}_{i_2+i_3,j_4;i_2,j_2}\varsigma^+_{i_1,j_1}\varsigma^-_{i_2+i_3,j_4}\Big)\\
&= \sum_{j_4\in I_{i_2+i_3},\ j_5\in I_{i_1+i_2+i_3}} \Big[(-1)^{i_3}\Big(a^{-,j_2}_{i_3,j_3;i_2+i_3,j_4}a^{+,j_5}_{i_1,j_1;i_2+i_3,j_4}\\
&\quad + a^{+,j_3}_{i_2+i_3,j_4;i_2,j_2}a^{-,j_1}_{i_2+i_3,j_4;i_1+i_2+i_3,j_5}\Big)\varsigma^+_{i_1+i_2+i_3,j_5}\\
&\quad + (-1)^{i_1+i_2}a^{+,j_3}_{i_2+i_3,j_4;i_2,j_2}a^{+,j_4}_{i_1+i_2+i_3,j_5;i_1,j_1}\varsigma^-_{i_1+i_2+i_3,j_5}\Big]
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
(\varsigma^+_{i_1,j_1}\varsigma^-_{i_2,j_2})\varsigma^+_{i_3,j_3} &= \sum_{j_4\in I_{i_1+i_2}} \Big((-1)^{i_2}a^{-,j_1}_{i_2,j_2;i_1+i_2,j_4}\varsigma^+_{i_1+i_2,j_4} + (-1)^{i_1}a^{+,j_2}_{i_1+i_2,j_4;i_1,j_1}\varsigma^-_{i_1+i_2,j_4}\Big)\varsigma^+_{i_3,j_3}\\
&= \sum_{j_4\in I_{i_1+i_2},\ j_5\in I_{i_1+i_2+i_3}} \Big[\Big((-1)^{i_2}a^{+,j_5}_{i_1+i_2,j_4;i_3,j_3}a^{-,j_1}_{i_2,j_2;i_1+i_2,j_4}\\
&\quad + (-1)^{i_1}a^{-,j_3}_{i_1+i_2+i_3,j_5;i_1+i_2,j_4}a^{+,j_2}_{i_1+i_2,j_4;i_1,j_1}\Big)\varsigma^+_{i_1+i_2+i_3,j_5}\\
&\quad + (-1)^{i_1}a^{+,j_4}_{i_3,j_3;i_1+i_2+i_3,j_5}a^{+,j_2}_{i_1+i_2,j_4;i_1,j_1}\varsigma^-_{i_1+i_2+i_3,j_5}\Big]
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
\varsigma^+_{i_1,j_1}\Big(\varsigma^-_{i_2,j_2}\varsigma^+_{i_3,j_3}\Big) &= \sum_{j_4\in I_{i_2+i_3}} \varsigma^+_{i_1,j_1}\Big(a^{+,j_2}_{i_3,j_3;i_2+i_3,j_4}\varsigma^-_{i_2+i_3,j_4} + a^{-,j_3}_{i_2+i_3,j_4;i_2,j_2}\varsigma^+_{i_2+i_3,j_4}\Big)\\
&= \sum_{j_4\in I_{i_1+i_2},\ j_5\in I_{i_1+i_2+i_3}} \Big[\Big((-1)^{i_2+i_3}a^{-,j_1}_{i_2+i_3,j_4;i_1+i_2+i_3,j_5}a^{+,j_2}_{i_3,j_3;i_2+i_3,j_4}\\
&\quad + a^{+,j_5}_{i_1,j_1;i_2+i_3,j_4}a^{-,j_3}_{i_2+i_3,j_4;i_2,j_2}\Big)\varsigma^+_{i_1+i_2+i_3,j_5}\\
&\quad + (-1)^{i_1}a^{+,j_4}_{i_1+i_2+i_3,j_5;i_1,j_1}a^{+,j_2}_{i_3,j_3;i_2+i_3,j_4}\varsigma^-_{i_1+i_2+i_3,j_5}\Big].
\end{aligned}
\tag{3.14}
$$

Thus we obtain

$$
\begin{aligned}
&\sum_{j_4\in I_{i_1+i_2}} a^{+,j_4}_{i_1,j_1;i_2,j_2}a^{-,j_4}_{i_3,j_3;i_1+i_2+i_3,j_5}\\
&\qquad = \sum_{j_4\in I_{i_2+i_3}} \Big(a^{-,j_2}_{i_3,j_3;i_2+i_3,j_4}a^{+,j_5}_{i_1,j_1;i_2+i_3,j_4} + a^{+,j_3}_{i_2+i_3,j_4;i_2,j_2}a^{-,j_1}_{i_2+i_3,j_4;i_1+i_2+i_3,j_5}\Big)
\end{aligned}
\tag{3.15}
$$

by (3.11) and (3.12), and

$$\sum_{j_4\in I_{i_1+i_2}} \Big((-1)^{i_2} a^{+,j_5}_{i_1+i_2,j_4;i_3,j_3} a^{-,j_1}_{i_2,j_2;i_1+i_2,j_4} + (-1)^{i_1} a^{-,j_3}_{i_1+i_2+i_3,j_5;i_1+i_2,j_4} a^{+,j_2}_{i_1+i_2,j_4;i_1,j_1}\Big)$$
$$= \sum_{j_4\in I_{i_1+i_2}} \Big((-1)^{i_2+i_3} a^{-,j_1}_{i_2+i_3,j_4;i_1+i_2+i_3,j_5} a^{+,j_2}_{i_3,j_3;i_2+i_3,j_4} + a^{+,j_5}_{i_1,j_1;i_2+i_3,j_4} a^{-,j_3}_{i_2+i_3,j_4;i_2,j_2}\Big) \tag{3.16}$$

by (3.13) and (3.14), for $i_p \in \mathbb{Z}_2,\ j_p \in I_{i_p}$ with $p = 1, 2, 3$ and $j_5 \in I_{i_1+i_2+i_3}$. Conversely, we have the following conclusion.

Proposition 3.1. *Suppose that we have two $\mathbb{Z}_2$-graded associative algebra operations $\circ_+$ and $\circ_-$ on a $\mathbb{Z}_2$-graded vector space $\mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1$ such that under a basis $\{\vartheta_{i,j} \mid i \in \mathbb{Z}_2,\ j \in I_i\}$ of $\mathcal{B}$, the structure constants*

$$\Big\{a^{\pm,j_3}_{i_1,j_1;i_2,j_2}\Big| i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in I_{i_1},\ j_2 \in I_{i_2},\ j_3 \in I_{i_1+i_2}\Big\}. \tag{3.17}$$

satisfy (3.15) and (3.16), where

$$\vartheta_{i_1,j_1} \circ_\pm \vartheta_{i_2,j_2} = \sum_{j_3\in I_{i_1+i_2}} a^{\pm,j_3}_{i_1,j_1;i_2,j_2} \vartheta_{i_1,i_2,j_3} \qquad \text{for} \quad i_1, i_2 \in \mathbb{Z}_2, j_1 \in I_{i_1}, j_2 \in I_{i_2}. \tag{3.18}$$

Let $\mathcal{A}^\pm_i$ be the vector spaces with a basis $\{\varsigma^\pm_{i,j} \mid j \in I_i\}$ for $i \in \mathbb{Z}_2$. Set

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 = \mathcal{A}^+ \oplus \mathcal{A}^- \tag{3.19}$$

with

$$\mathcal{A}_i = \mathcal{A}^+_i + \mathcal{A}^-_i \qquad \mathcal{A}^\pm = \mathcal{A}^\pm_0 \oplus \mathcal{A}^\pm_1. \tag{3.20}$$

We define the multiplication operation on $\mathcal{A}$ by (3.4), (3.9) and (3.10), and the bilinear form $\langle\cdot,\cdot\rangle$ by

$$\langle \mathcal{A}^\pm, \mathcal{A}^\pm\rangle = \{0\} \tag{3.21}$$

and

$$\langle \varsigma^-_{i_1,j_1}, \varsigma^+_{i_2,j_2}\rangle = (-1)^{i_1}\langle \varsigma^+_{i_2,j_2}, \varsigma^-_{i_1,j_1}\rangle = \delta_{i_1,i_2}\delta_{j_1,j_2} \tag{3.22}$$

for $i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in I_{i_1},\ j_2 \in I_{i_2}$. Then $\mathcal{A}$ forms a $\mathbb{Z}_2$-graded polarized associative algebra.

Proof. By (3.5)–(3.8), (3.11)–(3.14) and the symmetry of (3.15) and (3.16) with respect to the signs '+' and '–', we only need to verify

$$(\varsigma^+_{i_1,j_1}\varsigma^-_{i_2,j_2})\varsigma^-_{i_3,j_3} = \varsigma^+_{i_1,j_1}(\varsigma^-_{i_2,j_2}\varsigma^-_{i_3,j_3}) \tag{3.23}$$
$$\langle \varsigma^+_{i_1,j_1}\varsigma^-_{i_2,j_2}, \varsigma^+_{i_1+i_2,j_3}\rangle = \langle \varsigma^+_{i_1,j_1}, \varsigma^-_{i_2,j_2}\varsigma^+_{i_1+i_2,j_3}\rangle \tag{3.24}$$

for $i_p \in \mathbb{Z}_2$ and $j_p \in I_p$. Note

$$(\varsigma^+_{i_1,j_1}\varsigma^-_{i_2,j_2})\varsigma^-_{i_3,j_3} = \sum_{j_4\in I_{i_1+i_2}} \Big((-1)^{i_2} a^{-,j_1}_{i_2,j_2;i_1+i_2,j_4}\varsigma^+_{i_1+i_2,j_4} + (-1)^{i_1} a^{+,j_2}_{i_1+i_2,j_4;i_1,j_1}\varsigma^-_{i_1+i_2,j_4}\Big)\varsigma^-_{i_3,j_3}$$
$$= \sum_{j_4\in I_{i_1+i_2},\ j_5\in I_{i_1+i_2+i_3}} \Big((-1)^{i_2+i_3} a^{-,j_1}_{i_2,j_2;i_1+i_2,j_4} a^{-,j_4}_{i_3,j_3;i_1+i_2+i_3,j_5}\varsigma^+_{i_1+i_2+i_3,j_5}$$
$$+ (-1)^{i_1}\Big(a^{-,j_1}_{i_2,j_2;i_1+i_2,j_4} a^{+,j_3}_{i_1+i_2+i_3,j_5;i_1+i_2,j_4}$$
$$+ a^{-,j_5}_{i_1+i_2,j_4;i_3,j_3} a^{+,j_2}_{i_1+i_2,j_4;i_1,j_1}\Big)\Big)\varsigma^-_{i_1+i_2+i_3,j_5} \tag{3.25}$$

$$\varsigma^+_{i_1,j_1}(\varsigma^-_{i_2,j_2}\varsigma^-_{i_3,j_3}) = \sum_{j_4\in I_{i_2+i_3}} a^{-,j_4}_{i_2,j_2;i_3,j_3}\varsigma^+_{i_1,j_1}\varsigma^-_{i_2+i_3,j_4}$$
$$= \sum_{j_4\in I_{i_2+i_3},\ j_5\in I_{i_1+i_2+i_3}} a^{-,j_4}_{i_2,j_2;i_3,j_3}\Big((-1)^{i_2+i_3}a^{-,j_1}_{i_2+i_3,j_4;i_1+i_2+i_3,j_5}\varsigma^+_{i_1+i_2+i_3,j_5}$$
$$+ (-1)^{i_1}a^{+,j_4}_{i_1+i_2+i_3,j_5;i_1,j_1}\varsigma^-_{i_1+i_2+i_3,j_5}\Big). \tag{3.26}$$

So (3.23) follows from (3.25), (3.26) and (3.15) with the change of indices:

$$i_1 \to i_1+i_2+i_3 \to i_3 \to i_2 \to i_1 \qquad j_1 \to j_5 \to j_3 \to j_2 \to j_1. \tag{3.27}$$

Moreover,

$$\langle \varsigma^+_{i_1,j_1}\varsigma^-_{i_2,j_2}, \varsigma^+_{i_1+i_2,j_3}\rangle = (-1)^{i_1}a^{+,j_2}_{i_1+i_2,j_3;i_1,j_1} \tag{3.28}$$

by (3.9) and (3.22), and

$$\langle \varsigma^+_{i_1,j_1}, \varsigma^-_{i_2,j_2}\varsigma^+_{i_1+i_2,j_3}\rangle = (-1)^{i_1}a^{+,j_2}_{i_1+i_2,j_3;i_1,j_1} \tag{3.29}$$

by (3.10) and (3.22). Therefore, (3.24) holds. □

Let $\mathcal{A}$ be a polarized $\mathbb{Z}_2$-graded associative algebra with the bilinear form $\langle\cdot,\cdot\rangle_1$ and let $\mathcal{B}$ be a $\mathbb{Z}_2$-graded associative algebra with a nondegenerate supersymmetric associative bilinear form $\langle\cdot,\cdot\rangle_2$ (cf. (2.32)). Here $\mathcal{B}$ may not be polarized. Set

$$\tilde{\mathcal{A}}_0 = \mathcal{A}_0\otimes_{\mathbb{F}}\mathcal{B}_0 + \mathcal{A}_1\otimes_{\mathbb{F}}\mathcal{B}_1 \qquad \tilde{\mathcal{A}}_1 = \mathcal{A}_0\otimes_{\mathbb{F}}\mathcal{B}_1 + \mathcal{A}_1\otimes_{\mathbb{F}}\mathcal{B}_0 \tag{3.30}$$
$$\tilde{\mathcal{A}}^+ = \mathcal{A}^+\otimes_{\mathbb{F}}\mathcal{B} \qquad \tilde{\mathcal{A}}^- = \mathcal{A}^-\otimes_{\mathbb{F}}\mathcal{B} \tag{3.31}$$

and

$$\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_0\oplus\tilde{\mathcal{A}}_1 = \tilde{\mathcal{A}}^+\oplus\tilde{\mathcal{A}}^-. \tag{3.32}$$

Define the multiplication and bilinear form on $\tilde{\mathcal{A}}$ by

$$\begin{aligned}(a_1\otimes b_1)(a_2\otimes b_2) &= (-1)^{i_1,i_2}a_1a_2\otimes b_1b_2\\ \langle a_1\otimes b_1, a_2\otimes b_2\rangle &= (-1)^{i_1i_2}\langle a_1,a_2\rangle_1\langle b_1,b_2\rangle_2\end{aligned} \tag{3.33}$$

for $a_1\in\mathcal{A},\ a_2\in\mathcal{A}_{i_1}$ and $b_1\in\mathcal{B}_{i_2},\ b_2\in\mathcal{B}$. It is straightforward to verify the following proposition.

Proposition 3.2. *The space $\tilde{\mathcal{A}}$ forms a polarized $\mathbb{Z}_2$-graded associative algebra.*

Example 3.1. In the algebra $\mathbb{F}[t,t^{-1}]$ of Laurent polynomials, we define the bilinear form

$$\langle t^m, t^n\rangle = \delta_{m+n,-1} \qquad \text{for } m,n\in\mathbb{Z}. \tag{3.34}$$

Set

$$(\mathbb{F}[t,t^{-1}])^+ = \mathbb{F}[t] \qquad (\mathbb{F}[t,t^{-1}])^- = \mathbb{F}[t^{-1}]t^{-1}. \tag{3.35}$$

Then $\mathbb{F}[t,t^{-1}] = (\mathbb{F}[t,t^{-1}])^+\oplus(\mathbb{F}[t,t^{-1}])^-$ forms a polarized associative algebra (with $(\mathbb{F}[t,t^{-1}])_0 = \mathbb{F}[t,t^{-1}]$ and $(\mathbb{F}[t,t^{-1}])_1 = \{0\}$).

Example 3.2. Let k be a positive integer and let $k_1\in\overline{0,k-1}$. Denote by $M_{k\times k}(\mathbb{F})$ the algebra of $k\times k$ matrices with entries in $\mathbb{F}$, and by $E_{j,l}$ the matrix with 1 as its (j,l)-entry and 0 as the others. Define

$$M_{k\times k}(\mathbb{F})_0 = \sum_{j,l\in\overline{1,k_1}}\mathbb{F}E_{j,l} + \sum_{p,q\in\overline{k_1+1,k}}\mathbb{F}E_{p,q} \tag{3.36}$$

$$M_{k\times k}(\mathbb{F})_1 = \sum_{j\in\overline{1,k_1},\ p\in\overline{k_1+1,k}}(\mathbb{F}E_{j,p}+\mathbb{F}E_{p,j}) \tag{3.37}$$

and

$$\mathrm{Tr}\, A = \sum_{j\in\overline{1,k_1}} a_{j,j} - \sum_{p\in\overline{k_1+1,k}} a_{p,p} \qquad \text{for } A = \sum_{j,l=1}^{k} a_{j,l}E_{j,l} \in M_{k\times k}(\mathbb{F}). \tag{3.38}$$

Moreover, we define

$$\langle A, B\rangle = \mathrm{Tr}\, AB \qquad \text{for } A, B \in M_{k\times k}(\mathbb{F}). \tag{3.39}$$

Then $M_{k\times k}(\mathbb{F})$ forms a $\mathbb{Z}_2$-graded associative algebra with the supersymmetric nondegenerate associative bilinear form $\langle\cdot,\cdot\rangle$.

According to proposition 3.2,

$$M_{k\times k}(\mathbb{F}) \otimes_{\mathbb{F}} \mathbb{F}[t, t^{-1}] \cong M_{k\times k}(\mathbb{F}[t, t^{-1}]) \tag{3.40}$$

forms a polarized $\mathbb{Z}_2$-graded associative algebra, where $M_{k\times k}(\mathbb{F}[t, t^{-1}])$ denote the algebra of $k \times k$ matrices with entries in $\mathbb{F}[t, t^{-1}]$.

Example 3.3. Let $k > 1$ be integer and $\mathbb{F} = \mathbb{C}$ the field of complex numbers. The *Hecke algebra* $\mathcal{H}_k$ is an associative algebra generated by $\{T_1, \dots, T_{k-1}\}$ with the following defining relations

$$T_iT_j = T_jT_i \qquad \text{whenever } |i - j| \geqslant 2 \tag{3.41}$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad T_i^2 = (q-1)T_i + q \tag{3.42}$$

for $i, j \in \overline{1, k-1}$, where $0 \neq q \in \mathbb{C}$. Let $\zeta \in \mathbb{C}$ be a fixed constant. According to section 5 of [16], there exists a unique trace map 'Tr' of $\mathcal{H}_k$ such that

$$\mathrm{Tr}\,(1_{\mathcal{H}_k}) = 1 \qquad \mathrm{Tr}\,(aT_nb) = \zeta\,\mathrm{Tr}\,(ab) \qquad \text{for } a, b \in \mathcal{H}_n \tag{3.43}$$

with $n \in \overline{1, k-2}$. This trace map is the key to define the well-known 'Jones polynomials' of knots (e.g., cf. [16]). Furthermore, we define the bilinear form

$$\langle u, v\rangle = \mathrm{Tr}\, uv \qquad \text{for } u, v \in \mathcal{H}_k. \tag{3.44}$$

Then $\langle\cdot,\cdot\rangle$ is a nondegenerate associative symmetric bilinear form of $\mathcal{H}_k$ under a certain condition on ζ.

According to proposition 3.2 and (3.40),

$$\mathcal{H}_k \otimes_{\mathbb{F}} M_{k\times k}(\mathbb{F}[t, t^{-1}]) \tag{3.45}$$

forms a $\mathbb{Z}_2$-graded polarized associative algebras under a certain condition of ζ. Here we treat the odd part of $\mathcal{H}_k$ as zero.

Example 3.4. Let G be a group. Take a map $\varepsilon : G \times G \to \mathbb{F}^{\times} = \mathbb{F} \setminus \{0\}$ such that

$$\varepsilon(g_1, g_2)\varepsilon(g_1g_2, g_3) = \varepsilon(g_1, g_2g_3)\varepsilon(g_2, g_3) \qquad \text{for } g_1, g_2, g_3 \in G. \tag{3.46}$$

Let $\mathbb{F}[G]_\varepsilon$ be a vector space with a basis $\{u_g | g \in G\}$. Define the multiplication on $\mathbb{F}[G]_\varepsilon$ by

$$u_{g_1}u_{g_2} = \varepsilon(g_1, g_2)u_{g_1g_2} \qquad \text{for } g_1, g_2 \in G. \tag{3.47}$$

Then $\mathbb{F}[G]_\varepsilon$ forms an associative algebra, which is called a *twisted group algebra* of $\mathcal{G}$. Moreover, we define $\mathrm{Tr} : \mathbb{F}[G]_\varepsilon \to \mathbb{F}$ by

$$\mathrm{Tr}\,(u_g) = \delta_{g,0} \qquad \text{for } g \in G \tag{3.48}$$

and

$$\langle u, v\rangle = \mathrm{Tr}\,(uv) \qquad \text{for } u, v \in \mathbb{F}[G]_\varepsilon. \tag{3.49}$$

It is straightforward to verify that $\langle\cdot,\cdot\rangle$ is a nondegenerate symmetric associative bilinear form of $\mathbb{F}[G]_\varepsilon$.

According to proposition 3.2 and (3.45),

$$\mathbb{F}[G]_\varepsilon \otimes_{\mathbb{F}} \mathcal{H}_k \otimes_{\mathbb{F}} M_{k\times k}(\mathbb{F}[t, t^{-1}]) \tag{3.50}$$

forms a $\mathbb{Z}_2$-graded polarized associative algebra. Here we treat the odd part of $\mathbb{F}[G]_\varepsilon$ as zero.

The following work is necessary for defining Poisson superpairs over an infinite-dimensional $\mathbb{Z}_2$-graded polarized associative algebra.

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded polarized associative algebra with the bilinear form $\langle\cdot,\cdot\rangle$. Define

$$\mathcal{A}^{(0)} = \{u \in \mathcal{A}^+ \mid \mathcal{A}^- u \subset \mathcal{A}^-\} \tag{3.51}$$

and

$$\mathcal{A}^{(m+1)} = \{u \in \mathcal{A}^+ \mid \mathcal{A}^- u \subset \mathcal{A}^{(m)} + \mathcal{A}^-\} \tag{3.52}$$

for $m \in \mathbb{N}$ by induction. Then we have

$$\mathcal{A}^{(n)} \subset \mathcal{A}^{(n+1)} \qquad \text{for } n \in \mathbb{N}. \tag{3.53}$$

Set

$$\mathcal{A}^{(-1)} = \mathcal{A}^- \qquad \mathcal{A}^{(-2-m)} = \{u \in \mathcal{A}^- \mid \langle u, \mathcal{A}^{(m)}\rangle = \{0\}\} \qquad \text{for } m \in N. \tag{3.54}$$

Then (3.53) also holds for a negative intger $n < -1$. Since $\mathcal{A}^+$ and $\mathcal{A}^-$ are $\mathbb{Z}_2$-graded subalgebras of $\mathcal{A}$, all $\mathcal{A}^{(n)}$ with $n \in \mathbb{Z}$ are $\mathbb{Z}_2$-graded subspaces of $\mathcal{A}$ by (2.32).

Proposition 3.3. *For $m, n \in \mathbb{N}$,*

$$\mathcal{A}^{(m)}\mathcal{A}^{(n)} \subset \mathcal{A}^{(m+n)} \tag{3.55}$$

$$\mathcal{A}^{(m+n)}\mathcal{A}^{(-n-1)} \qquad \mathcal{A}^{(-n-1)}\mathcal{A}^{(m+n)} \subset \mathcal{A}^{(m-1)} + \mathcal{A}^- \tag{3.56}$$

$$\mathcal{A}^{(m)}\mathcal{A}^{(-m-n-2)} \qquad \mathcal{A}^{(-m-n-2)}\mathcal{A}^{(m)} \subset \mathcal{A}^{(-n-2)} \tag{3.57}$$

$$\mathcal{A}^{(-m-1)}\mathcal{A}^{(-n-1)} \subset \mathcal{A}^{(-m-n-2)} \tag{3.58}$$

Proof. We prove (3.55) by induction on m. Note

$$\mathcal{A}^-(\mathcal{A}^{(0)}\mathcal{A}^{(n)}) = (\mathcal{A}^-\mathcal{A}^{(0)})\mathcal{A}^{(n)} \subset \mathcal{A}^-\mathcal{A}^{(n)} \subset \mathcal{A}^{(n-1)} + \mathcal{A}^- \tag{3.59}$$

by (3.51) and (3.52). Moreover, (3.52) and (3.59) imply (3.55) with $m = 0$. Suppose that (3.55) holds for $m =$ k with $k \in \mathbb{N}$. We have

$$\mathcal{A}^-(\mathcal{A}^{(k+1)}\mathcal{A}^{(n)}) = (\mathcal{A}^-\mathcal{A}^{(k+1)})\mathcal{A}^{(n)} \subset (\mathcal{A}^{(k)} + \mathcal{A}^-)\mathcal{A}^{(n)} \subset \mathcal{A}^{(k+n)} + \mathcal{A}^- \tag{3.60}$$

by (3.52). Again (3.52) and (3.60) imply (3.55) with $m =$ k +1. So (3.55) holds.

We define

$$\mathcal{A}^{[0]} = \{u \in \mathcal{A}^+ \mid u\mathcal{A}^- \subset \mathcal{A}^-\} \tag{3.61}$$

and

$$\mathcal{A}^{[m+1]} = \{u \in \mathcal{A}^+ \mid u\mathcal{A}^- \subset \mathcal{A}^{[m]} + \mathcal{A}^-\} \tag{3.62}$$

for $m \in \mathbb{N}$ by induction. Note

$$\langle \mathcal{A}^-, \mathcal{A}^{(0)}\mathcal{A}^-\rangle = \langle \mathcal{A}^-\mathcal{A}^{(0)}, \mathcal{A}^-\rangle \subset \langle \mathcal{A}^-, \mathcal{A}^-\rangle = \{0\} \tag{3.63}$$

by (2.32) and (2.33). By the nondegeneracy of $\langle\cdot,\cdot\rangle$ and (2.34), we have

$$\mathcal{A}^{(0)}\mathcal{A}^- \subset \mathcal{A}^-. \tag{3.64}$$

Thus

$$\mathcal{A}^{(0)} \subset \mathcal{A}^{[0]}. \tag{3.65}$$

By the symmetric proof as that in the above, we also have $\mathcal{A}^{[0]} \subset \mathcal{A}^{(0)}$. Hence

$$\mathcal{A}^{(0)} = \mathcal{A}^{[0]}. \tag{3.66}$$

Suppose that

$$\mathcal{A}^{(m)} = \mathcal{A}^{[m]} \tag{3.67}$$

for $m \leqslant k$ with $k \in \mathbb{N}$. We have

$$\begin{aligned}\mathcal{A}^-(\mathcal{A}^{(k+1)}\mathcal{A}^-) &= (\mathcal{A}^-\mathcal{A}^{(k+1)})\mathcal{A}^- \subset (\mathcal{A}^{(k)} + \mathcal{A}^-)\mathcal{A}^- \subset \mathcal{A}^{[k]}\mathcal{A}^- + \mathcal{A}^- \subset \mathcal{A}^{[k-1]} + \mathcal{A}^- \\ &= \mathcal{A}^{(k-1)} + \mathcal{A}^-\end{aligned} \tag{3.68}$$

by (3.52), (3.62) and (3.67). Definition (3.52) implies

$$\mathcal{A}^{(k+1)}\mathcal{A}^- \subset \mathcal{A}^{(k)} = \mathcal{A}^{[k]}. \tag{3.69}$$

By (3.62), we have

$$\mathcal{A}^{(k+1)} \subset \mathcal{A}^{[k+1]}. \tag{3.70}$$

A symmetric argument shows $\mathcal{A}^{[k+1]} \subset \mathcal{A}^{(k+1)}$. Hence

$$\mathcal{A}^{(k+1)} = \mathcal{A}^{[k+1]}. \tag{3.71}$$

Therefore, (3.67) holds for any $m \in \mathbb{N}$ by induction.

Expressions (3.52), (3.62) and (3.67) imply that (3.56) holds for $n = 0$. Assume $n > 0$. Then we have

$$\langle \mathcal{A}^-, \mathcal{A}^{(n)}\mathcal{A}^{(-n-1)}\rangle \subset \langle \mathcal{A}^-\mathcal{A}^{(n)}, \mathcal{A}^{(-n-1)}\rangle \subset \langle \mathcal{A}^{(n-1)} + \mathcal{A}^-, \mathcal{A}^{(-n-1)}\rangle = \{0\} \tag{3.72}$$

by (2.32), (2.33) and (3.54). The nondegeneracy of $\langle \cdot, \cdot \rangle$ and (2.34) imply

$$\mathcal{A}^{(n)}\mathcal{A}^{(-n-1)} \subset \mathcal{A}^- = \mathcal{A}^{(-1)}. \tag{3.73}$$

Suppose that

$$\mathcal{A}^{(m+n)}\mathcal{A}^{(-n-1)} \subset \mathcal{A}^{(m-1)} + \mathcal{A}^- \tag{3.74}$$

holds for some $m \in \mathbb{N}$. We have

$$\begin{aligned}\mathcal{A}^-(\mathcal{A}^{(m+1+n)}(\mathcal{A}^{(-n-1)}) &= (\mathcal{A}^-\mathcal{A}^{(m+1+n)})\mathcal{A}^{(-n-1)} \subset (\mathcal{A}^{(m+n)} + \mathcal{A}^-)\mathcal{A}^{(-n-1)} \\ &\subset \mathcal{A}^{(m+n)}\mathcal{A}^{(-n-1)} + \mathcal{A}^- \subset \mathcal{A}^{(m-1)} + \mathcal{A}^-\end{aligned} \tag{3.75}$$

by (3.52), (3.74) and the fact $\mathcal{A}^-$ is a subalgebra of A (note $\mathcal{A}^{(-n-1)} \subset \mathcal{A}^-$ by (3.54)). Again the definition (3.52) imply

$$\mathcal{A}^{(m+1+n)}\mathcal{A}^{(-n-1)} \subset \mathcal{A}^{(m)} + \mathcal{A}^-. \tag{3.76}$$

By induction, (3.74) holds for any $m \in \mathbb{N}$. By means of (3.67), we can symmetrically prove

$$\mathcal{A}^{(-n-1)}\mathcal{A}^{(m+n)} \subset \mathcal{A}^{(m-1)} + \mathcal{A}^-. \tag{3.77}$$

This proves (3.56).

For $m, n \in \mathbb{N}$, we have

$$\langle \mathcal{A}^{(n)}, \mathcal{A}^{(m)}\mathcal{A}^{(-m-n-2)}\rangle = \langle \mathcal{A}^{(n)}\mathcal{A}^{(m)}, \mathcal{A}^{(-m-n-2)}\rangle \subset \langle \mathcal{A}^{(m+n)}, \mathcal{A}^{(-m-n-2)}\rangle = \{0\} \tag{3.78}$$

by (2.32), (3.54) and (3.55). Moreover, (3.54) and (3.78) imply

$$\mathcal{A}^{(m)}\mathcal{A}^{(-m-n-2)} \subset \mathcal{A}^{(-n-2)}. \tag{3.79}$$

Symmetrically, we can prove

$$\mathcal{A}^{(-m-n-2)}\mathcal{A}^{(m)} \subset \mathcal{A}^{(-n-2)} \tag{3.80}$$

by (3.67). So (3.57) holds.

Observe that

$$\langle \mathcal{A}^{(m+n)}, \mathcal{A}^{(-m-1)}\mathcal{A}^{(-n-1)}\rangle = \langle \mathcal{A}^{(m+n)}\mathcal{A}^{(-m-1)}, \mathcal{A}^{(-n-1)} \subset \langle \mathcal{A}^{(n-1)} + \mathcal{A}^{-}, \mathcal{A}^{(-n-1)}\rangle = \{0\} \tag{3.81}$$

by (2.32), (2.33), (3.54) and (3.56). Moreover, (3.54) and (3.81) imply

$$\mathcal{A}^{(-m-1)}\mathcal{A}^{(-n-1)} \subset \mathcal{A}^{(-m-n-2)} \tag{3.82}$$

that is, (3.58) holds. □

4. Poisson superpairs

In this section, we shall construct Poisson superpairs over a $\mathbb{Z}_2$-graded polarized associative algebra.

Let Λ be a vector space, not necessarily finite-dimensional. Let $F(\Lambda)$ be the free associative algebra generated by Λ. Then the exterior algebra $\mathcal{E}$ generated by Λ is isomorphic to

$$\mathcal{E} = F(\Lambda)/(\{uv + vu \mid u, v \in \Lambda\}). \tag{4.1}$$

We can identify Λ with its image in $\mathcal{E}$. Note that

$$\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \qquad \text{where } \mathcal{E}_0 = \sum_{n=0}^{\infty} \Lambda^{2n}, \quad \mathcal{E}_1 = \sum_{n=0}^{\infty} \Lambda^{2n+1}. \tag{4.2}$$

With respect to the above grading, $\mathcal{E}$ becomes a super commutative associative algebra, that is,

$$uv = (-1)^{ij} vu \qquad \text{for } u \in \mathcal{E}_i,\ v \in \mathcal{E}_j. \tag{4.3}$$

Let

$$\mathcal{A} = \mathcal{A}^{+} \oplus \mathcal{A}^{-} = \mathcal{A}_0 \oplus \mathcal{A}_1 \tag{4.4}$$

be infinite-dimensional $\mathbb{Z}_2$-graded polarized associative algebra with the bilinear form $\langle \cdot, \cdot \rangle$. Recall the notations of $\mathcal{A}^{(n)}$ with $n \in \mathbb{Z}$ defined in (3.51), (3.52) and (3.54). Their properties, which are important to the following construction, have been presented in proposition 3.3 (cf. (3.55)–(3.58)). Assume that the algebra $\mathcal{A}$ satisfies the following condition:

$$\mathcal{A}^{+} = \bigcup_{m=0}^{\infty} \mathcal{A}^{(m)} \qquad \dim \mathcal{A}^{(m)} < \infty. \tag{4.5}$$

We call such an algebra $\mathcal{A}$ a *semi-finitely-filtered polarized $\mathbb{Z}_2$-graded associative algebra*. Examples of this type of algebras have been given in examples 3.1–3.4.

Set

$$\mathcal{A}_i^{(m)} = \mathcal{A}^{(m)} \bigcap \mathcal{A}_i \qquad k_{i,m} = \dim \mathcal{A}_i^{(m)} \tag{4.6}$$

for $i \in \mathbb{Z}_2$ and $m \in \mathbb{N}$. Take a basis $\{\varsigma_{i,j} \mid j \in J_i\}$ of $\mathcal{A}_i^{+}$ for $i \in \mathbb{Z}_2$ with

$$J_i = \mathbb{N} + 1 \ \text{ or } \ \overline{1, n} \qquad \text{for some } n \in \mathbb{N} + 1 \tag{4.7}$$

(which is guaranteed by (4.5)) such that

$$\{\varsigma_{i,j} \mid j \in \overline{1, k_{i,m}}\} \qquad \text{is a basis of } \mathcal{A}_i^{(m)} \tag{4.8}$$

for $i \in \mathbb{Z}_2$ and $m \in \mathbb{N}$.

Take a subset $\{\varsigma_{i,-j} \mid j \in J_i\}$ of $\mathcal{A}_i^-$ for $i \in \mathbb{Z}_2$ such that

$$\langle \varsigma_{i_1,-j_1}, \varsigma_{i_2,j_2}\rangle = \delta_{i_1,i_2}\delta_{j_1,j_2} \qquad \text{for } i_1, i_2 \in \mathbb{Z}_2,\ j_1 \in J_{i_1},\ j_2 \in J_{i_2} \tag{4.9}$$

and for any $u \in \mathcal{A}_i^-$,

$$u = \sum_{i\in Z_2,\ j\in J_i} \lambda_{i,j}\varsigma_{i,-j} \qquad \text{with } \lambda_{i,j} \in \mathbb{F}. \tag{4.10}$$

So $\{\varsigma_{i,-j} \mid j \in J_i\}$ is a *dual basis* in $\mathcal{A}_i^-$ of $\{\varsigma_{i,j} \mid j \in J_i\}$, in the possible sense of toplogical completion. By (3.54)

$$\varsigma_{i,-j} \in \mathcal{A}^{(-m-2)} \qquad \text{for } i \in \mathbb{Z}_2,\ k_{i,m} < j \in J_i. \tag{4.11}$$

Pick a positive integer ι. Set

$$I_i = (-J_i)\bigcap \overline{1, k_{i,\iota}} \qquad \text{for } i \in \mathbb{Z}_2. \tag{4.12}$$

For $i \in \mathbb{Z}_2$, let $\{x_{i,j} \mid j \in I_i\}$ be the variables taking values in $\mathcal{E}_i$ (cf. (4.2)). Denote by $\mathcal{P}$ the algebra of supersymmetric polynomials in $\{x_{i,j} \mid i \in \mathbb{Z}_2,\ j \in I_i\}$. Then $\mathcal{P}$ has the $\mathbb{Z}_2$-grading

$$\mathcal{P}_i = \text{Span}\left\{x_{i_1,j_1}\cdots x_{i_\ell,j_\ell} \mid \ell \in \mathbb{N},\ i_p \in \mathbb{Z}_2,\ j_p \in I_{i_p},\ \sum_{r=1}^{\ell} i_r \equiv i\right\} \tag{4.13}$$

for $i \in \mathbb{Z}_2$, and

$$fg = (-1)^{i_1 i_2} fg \qquad \text{for } f \in \mathcal{P}_{i_1},\ g \in \mathcal{P}_{i_2}. \tag{4.14}$$

For $i_1 \in \mathbb{Z}_2$ and $j_1 \in I_{i_1}$, we define a linear transformation ∂_{i_1,j_1} on $\mathcal{P}$ by

$$\partial_{i_1,j_1}(fg) = \partial_{i_1,j_1}(f)g + (-1)^{i_1 i_2} f\partial_{i_1,j_1}(g) \qquad \text{for } i_2 \in \mathbb{Z}_2,\ f \in \mathcal{P}_{i_2},\ g \in \mathcal{P} \tag{4.15}$$

and

$$\partial_{i_1,j_1}(x_{i_2,j_2}) = \delta_{i_1,i_2}\delta_{j_1,j_2} \qquad \text{for } i_2 \in \mathbb{Z}_2,\ j_{i_2} \in I_{i_2}. \tag{4.16}$$

Then ∂_{i_1,j_1} is a supersymmetric derivation of $\mathcal{P}$ with parity ι_1. Set

$$\begin{aligned} \mathcal{W}_0 &= \left\{\sum_{i\in Z_2,\ j\in I_i} f_{i,j}\partial_{i,j} \mid f_{i,j} \in \mathcal{P}_i\right\} \\ \mathcal{W}_1 &= \left\{\sum_{i\in Z_2,\ j\in I_i} f_{i,j}\partial_{i,j} \mid f_{i,j} \in \mathcal{P}_{i+1}\right\} \end{aligned} \tag{4.17}$$

and

$$\mathcal{W} = \mathcal{W}_0 + \mathcal{W}_1 \tag{4.18}$$

as a subspace of supersymmetric derivations of $\mathcal{P}$. The Lie superbracket on $\mathcal{W}$ is defined by

$$[d_1, d_2] = d_1 d_2 - (-1)^{i_1 i_2} d_2 d_1 \qquad \text{for } d_1 \in \mathcal{W}_{i_1},\ d_2 \in \mathcal{W}_{i_2}. \tag{4.19}$$

In fact, if $d_1 = \sum_{i\in Z_2,\ j\in I_i} f_{i,j}\partial_{i,j} \in \mathcal{W}_{i_1}$ and $d_2 = \sum_{i\in Z_2,\ j\in I_i} g_{i,j}\partial_{i,j} \in \mathcal{W}_{i_2}$, then we have

$$[d_1, d_2] = \sum_{i\in Z_2,\ j\in I_i}\ \sum_{i_1\in Z_2,\ j_1\in I_{i_1}} [f_{i_1,j_1}\partial_{i_1,j_1}(g_{i,j}) - (-1)^{i_1 i_2} g_{i_1,j_1}\partial_{i_1,j_1}(f_{i,j})]\partial_{i,j}. \tag{4.20}$$

Define the vector space

$$\overline{\mathcal{G}} = \left\{\sum_{i\in Z_2,\ j\in J_i} \xi_{i,-j}\varsigma_{i,-j} \mid \xi_{i,-j} \in \mathcal{P}\right\} + \sum_{i\in Z_2,\ j\in J_i} \mathcal{P}\varsigma_{i,j}. \tag{4.21}$$

Equip $\overline{\mathcal{G}}$ with the $\mathbb{Z}_2$-grading:

$$\overline{\mathcal{G}}_0 = \left\{ \sum_{i\in Z_2,\ j\in J_i} \xi_{i,-j}\varsigma_{i,-j} \mid \xi_{i,-j} \in \mathcal{P}_i \right\} + \sum_{i\in Z_2,\ j\in J_i} \mathcal{P}_i\varsigma_{i,j}, \tag{4.22}$$

$$\overline{\mathcal{G}}_1 = \left\{ \sum_{i\in Z_2,\ j\in J_i} \xi_{i,-j}\varsigma_{i,-j} \mid \xi_{i,-j} \in \mathcal{P}_{i+1} \right\} + \sum_{i\in Z_2,\ j\in J_i} \mathcal{P}_{i+1}\varsigma_{i,j}. \tag{4.23}$$

Set

$$\overline{J}_i = J_i \bigcup (-J_i) \qquad \text{for } i \in \mathbb{Z}_2. \tag{4.24}$$

For covenience, we use the notation

$$\varsigma_\xi = \sum_{i\in Z_2,\ j\in \overline{J}_i} \xi_{i,j}\varsigma_{i,j} \in \overline{\mathcal{G}}. \tag{4.25}$$

Moreover, we define the multiplication on $\overline{\mathcal{G}}$ by

$$\varsigma_\xi\varsigma_\eta = \sum_{i_1,i_2\in Z_2,\ j_1\in\overline{J}_{i_1},\ j_2\in\overline{J}_{i_2}} (-1)^{i_1(i_2+p)}\xi_{i_1,j_1}\eta_{i_2,j_2}\varsigma_{i_1,j_1}\varsigma_{i_2,j_2} \tag{4.26}$$

for $\varsigma_\xi \in \overline{\mathcal{G}}$ and $\varsigma_\eta \in \overline{\mathcal{G}}_p$. The above expression is well defined because of proposition 3.3. It can be verified that the space $\overline{\mathcal{G}}$ forms a $\mathbb{Z}_2$-graded associative algebra with respect to the multiplication in (4.26). Furthermore, we define a bilinear form $\overline{\mathcal{G}}$ by

$$\langle \varsigma_\xi, \varsigma_\eta\rangle = \sum_{i\in Z_2,\ j\in J_i} (-1)^{i(i+p)}(\xi_{i,-j}\eta_{i,j} + (-1)^i\xi_{i,j}\eta_{i,-j}) \tag{4.27}$$

for $\varsigma_\xi \in \overline{\mathcal{G}}$ and $\varsigma_\eta \in \overline{\mathcal{G}}_p$, where the sum is finite by (4.21). It is straightforward to verify that the above bilinear form is a $\mathbb{Z}_2$-graded supersymmetric associative bilinear form of $\overline{\mathcal{G}}$. In fact, one can view $\overline{\mathcal{G}}$ as an extension algebra of $\mathcal{A}$ with extended $\mathbb{Z}_2$-graded supersymmetric associative bilinear form $\langle\cdot,\cdot\rangle$.

Define

$$\mathcal{G} = \left\{ \sum_{i\in Z_2,\ j\in I_i} \xi_{i,j}\varsigma_{i,j} \mid \xi_{i,j} \in \mathcal{P} \right\} \tag{4.28}$$

(cf. (4.12)). Then $\mathcal{G}$ forms a $\mathbb{Z}_2$-graded subspace of $\overline{\mathcal{G}}$, that is,

$$\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1 \qquad \mathcal{G}_i = \mathcal{G} \bigcap \overline{\mathcal{G}}_i. \tag{4.29}$$

In general, $\mathcal{G}$ does not form a subalgebra of $\overline{\mathcal{G}}$. We shall use the convention that

$$\varsigma_\xi \in \mathcal{G} \text{ implies } \xi_{i,j} = 0 \qquad \text{for } j > k_{i,\iota}. \tag{4.30}$$

Thus

$$\varsigma_\xi = \sum_{i\in Z_2,\ j\in I_i} \xi_{i,j}\varsigma_{i,j} \qquad \text{if } \varsigma_\xi \in \mathcal{G} \tag{4.31}$$

(cf. (4.12)).

We define

$$\partial_{\varsigma_\xi} = \sum_{i\in Z_2,\ j\in I_i} \xi_{i,j}\partial_{i,j} \qquad \text{for } \varsigma_\xi \in \mathcal{G}. \tag{4.32}$$

The map

$$u \to \partial_u \tag{4.33}$$

gives a $\mathbb{Z}_2$-graded linear isomorphism between $\mathcal{G}$ and $\mathcal{W}$. Moreover, we define the action of $\mathcal{W}$ on $\mathcal{G}$ by

$$d(\varsigma_\xi) = \sum_{i\in Z_2,\ j\in I_i} d(\xi_{i,j})\varsigma_{i,j} \qquad \text{for } d \in \mathcal{W},\quad \varsigma_\xi \in \mathcal{G}. \tag{4.34}$$

We can use (4.33) as an identification of $\mathcal{G}$ with $\mathcal{W}$ (cf. (4.18) and (4.20)). We denote by $[\cdot,\cdot]_0$ the corresponding Lie superbracket of $\mathcal{G}$. By (4.19), (4.20) and (4.33), we have

$$[u,v]_0 = \partial_u(v) - (-1)^{i_1 i_2}\partial_v(u) \qquad \text{for } u \in \mathcal{G}_{i_1},\ v \in \mathcal{G}_{i_2}. \tag{4.35}$$

The algebra $\mathcal{P}$ becomes a $\mathcal{G}$-module with the action

$$u(f) = \partial_u(f) \qquad \text{for } u \in \mathcal{G},\ f \in \mathcal{P}. \tag{4.36}$$

Set

$$\Omega = \sum_{i\in Z_2,\ j\in I_i} \mathcal{P}\varsigma_{i,-j}. \tag{4.37}$$

Then Ω forms a $\mathbb{Z}_2$-graded subspace of $\overline{\mathcal{G}}$, that is,

$$\Omega = \Omega_0 \oplus \Omega_1 \qquad \Omega_i = \mathcal{G}\bigcap \overline{\mathcal{G}}_i. \tag{4.38}$$

We shall also use the convention that

$$\varsigma_\xi \in \Omega \ \text{ implies } \ \xi_{i,j} = 0 \qquad \text{for } j < -k_{i,\iota}. \tag{4.39}$$

Thus

$$\varsigma_\xi = \sum_{i\in Z_2,\ j\in I_i} \xi_{i,-j}\varsigma_{i,-j} \qquad \text{if } \varsigma_\xi \in \Omega, \tag{4.40}$$

where the sum is finite by (4.37). We identify Ω with a subspace of one-forms by

$$w(u) = \langle u, w\rangle \qquad \text{for } w \in \Omega,\ u \in \mathcal{G} \tag{4.41}$$

(cf. (4.27)). For $f = f_0 + f_1$ with $f_0 \in \mathcal{P}_0$ and $f_1 \in \mathcal{P}_1$, we define

$$\varsigma_{(f)} = \sum_{i\in Z_2,\ j\in J_i} ((-1)^i\partial_{i,-j}(f_0) + \partial_{i,-j}(f_1))\varsigma_{i,j} + \sum_{i\in Z_2,\ j\in\overline{1,k_{i,\iota}}} (\partial_{i,j}(f_0) + (-1)^i\partial_{i,j}(f_1))\varsigma_{i,-j} \in \Omega. \tag{4.42}$$

Then

$$df(u) = \partial_u(f) = \langle u, \varsigma_{(f)}\rangle \qquad \text{for } u \in \mathcal{G},\ f \in \mathcal{P}. \tag{4.43}$$

Hence

$$df = \varsigma_{(f)} \qquad \text{for } f \in \mathcal{P}. \tag{4.44}$$

Let $H : \Omega \to \mathcal{G}$ be a map of the form

$$H(\varsigma_\eta) = \sum_{i\in\mathbb{Z}_2,\ j\in I_i} \eta_{i,j}a_{i,j} \tag{4.45}$$

with

$$a_{i,j} \in \mathcal{G}_i \tag{4.46}$$

for $\varsigma_\eta \in \Omega$. Given $u \in \mathcal{G}_q$, we define the map $\partial_u(H) : \Omega \to \mathcal{G}$ by

$$\partial_u(H)(\varsigma_\eta) = \sum_{i_1,i_2\in\mathbb{Z}_2,\ j_1\in I_{i_1},\ j_2\in I_{i_2}} (-1)^{(i_2+p)q} \eta_{i_2,j_2} \partial_u(a_{i_2,j_2}) \tag{4.47}$$

for $\varsigma_\eta \in \Omega_p$. Note that for $\varsigma_\eta \in \Omega_p$ and $u \in \mathcal{G}_q$, we have

$$\partial_u(H(\varsigma_\eta)) = \partial_u(H)(\varsigma_\eta) + H(\partial_u(\varsigma_\eta)) \tag{4.48}$$

by (4.32), (4.34) and (4.47). Define

$$\{f, g\}_H = \langle H(\varsigma_{(f)}), \varsigma_{(g)}\rangle \qquad \text{for } f, g \in \mathcal{P}. \tag{4.49}$$

By (3.59)–(3.71) in [22] (also see section 4 in [20]) and (4.48), we have the following lemma.

Lemma 4.1. *The map* $\{\cdot,\cdot\}_H$ *forms a Poisson superbracket if H satisfies*

$$\langle H(v), u\rangle = -(-1)^{i_1 i_2}\langle H(u), v\rangle \tag{4.50}$$

and

$$\langle \partial_{H(u)}(H)(v), w\rangle + (-1)^{i_1(i_2+i_3)}\langle \partial_{H(v)}(H)(w), u\rangle + (-1)^{(i_1+i_2)i_3}\langle \partial_{H(w)}(H)(u), v\rangle = 0 \tag{4.51}$$

for $u \in \Omega_{i_1},\ v \in \Omega_{i_2}$ *and* $w \in \Omega_{i_3}$.

Denote

$$(\varsigma_\xi)_\pm = \sum_{i\in Z_2,\ j\in J_i} \xi_{i,\pm j}\, \varsigma_{i,\pm j} \qquad \text{for } \varsigma_\xi \in \overline{\mathcal{G}}. \tag{4.52}$$

Set

$$\overline{\mathcal{G}}^{\pm} = \{u_\pm \mid u \in \overline{\mathcal{G}}\}. \tag{4.53}$$

Then $\overline{\mathcal{G}}^{+}$ and $\overline{\mathcal{G}}^{-}$ form $\mathbb{Z}_2$-graded associative subalgebras of $\overline{\mathcal{G}}$. In fact,

$$\overline{\mathcal{G}} = \overline{\mathcal{G}}^{+} \oplus \overline{\mathcal{G}}^{-} \quad \text{is a polarization of } \overline{\mathcal{G}} \tag{4.54}$$

with respect to the multiplication in (4.26) and the bilinear form in (4.27). For any $v \in \overline{\mathcal{G}}$, we write

$$v = v_+ + v_- \qquad \text{with } v_\pm \in \overline{\mathcal{G}}^{\pm}. \tag{4.55}$$

In order to prove our main theorem in this section, we need the following lemma.

Lemma 4.2. *For* $u \in \overline{\mathcal{G}}_{i_1},\ v \in \overline{\mathcal{G}}_{i_2}$ *and* $w \in \overline{\mathcal{G}}_{i_3}$, *we have:*

$$\begin{aligned}\langle u, vw\rangle &= \langle u, v_+w_-\rangle + (-1)^{i_1(i_2+i_3)}\langle v, w_+u_-\rangle + (-1)^{(i_1+i_2)i_3}\langle w, u_+v_-\rangle \\ &= \langle u, v_-w^+\rangle + (-1)^{i_1(i_2+i_3)}\langle v, w_-u_+\rangle + (-1)^{(i_1+i_2)i_3}\langle w, u_-v_+\rangle.\end{aligned} \tag{4.56}$$

Proof. For an expression $h(u,v,w)$, we denote

$$h(u, v, w) + \text{c.p.} = h(u, v, w) + (-1)^{i_1(i_2+i_3)}h(v, w, u) + (-1)^{i_3(i_1+i_2)}h(w, u, v). \tag{4.57}$$

Moreover,

$$\langle uv, w\rangle = (-1)^{i_3(i_1+i_2)}\langle w, uv\rangle = (-1)^{i_3(i_1+i_2)}\langle wu, v\rangle = (-1)^{i_1(i_2+i_3)}\langle v, wu\rangle \tag{4.58}$$

by the supersymmetry and associativity of $\langle\cdot,\cdot\rangle$ on $\overline{\mathcal{G}}$.

Note by (2.33), (2.34), (4.58), and the supersymmetry and associativity of $\langle\cdot,\cdot\rangle$ on $\overline{\mathcal{G}}$,

$$\begin{aligned}\langle u, v_+w_-\rangle + \text{c.p.} &= \langle u_+ + u_-, v_+w_-\rangle + \text{c.p.}\\ &= \langle u_+, v_+w_-\rangle + \langle u_-, v_+w_-\rangle + \text{c.p.}\\ &= \langle u_+v_+, w_-\rangle + (-1)^{i_3(i_1+i_2)}\langle w_-u_-, v_+\rangle + \text{c.p.}\\ &= \langle u_+v_+, w\rangle + (-1)^{i_3(i_1+i_2)}\langle w_-u_-, v\rangle + \text{c.p.}\\ &= \langle u_+v_+, w\rangle + \langle u_-v_-, w\rangle + \text{c.p.}\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}\langle u, v_-w_+\rangle + \text{c.p.} &= \langle u_+ + u_-, v_-w_+\rangle + \text{c.p.}\\ &= \langle u_+, v_-w_+\rangle + \langle u_-, v_-w_+\rangle + \text{c.p.}\\ &= (-1)^{i_1(i_2+i_3)}\langle v_-, w_+u_+\rangle + \langle u_-v_-, w_+\rangle + \text{c.p.}\\ &= (-1)^{i_1(i_2+i_3)}\langle v, w_+u_+\rangle + \langle u_-v_-, w\rangle + \text{c.p.}\\ &= \langle u_+v_+, w\rangle + \langle u_-v_-, w\rangle + \text{c.p.}\end{aligned} \tag{4.60}$$

Thus

$$\begin{aligned}&\langle u, v_+w_-\rangle + (-1)^{i_1(i_2+i_3)}\langle v, w_+u_-\rangle + (-1)^{(i_1+i_2)i_3}\langle w, u_+v_-\rangle\\ &\quad= \langle u, v_-w_+\rangle + (-1)^{i_1(i_2+i_3)}\langle v, w_-u_+\rangle + (-1)^{(i_1+i_2)i_3}\langle w, u_-v_+\rangle\\ &\quad= \tfrac{1}{3}[\langle u, v_+w_-\rangle + \langle u, v_-w_+\rangle + \langle u_+v_+, w\rangle + \langle u_-v_-, w\rangle + \text{c.p.}]\\ &\quad= \tfrac{1}{3}[\langle u, v_+w_-\rangle + \langle u, v_-w_+\rangle + \langle u, v_+w_+\rangle + \langle u, v_-w_-\rangle + \text{c.p.}]\\ &\quad= \tfrac{1}{3}[\langle u, vw\rangle + \text{c.p.}]\\ &\quad= \langle u, vw\rangle\end{aligned} \tag{4.61}$$

by the supersymmetry and associativity of $\langle\cdot,\cdot\rangle$ on $\overline{\mathcal{G}}$. □

Define

$$[u, v] = u\cdot v - (-1)^{i_1i_2}v\cdot u \qquad \text{for } u\in\overline{\mathcal{G}}_i,\ v\in\overline{\mathcal{G}}_{i_2}. \tag{4.62}$$

Pick any

$$L_0 \in \mathcal{A}_0^- + \mathcal{A}_0^{(\iota+1)} \tag{4.63}$$

(cf. (3.51) and (3.52)) and set

$$L = L_0 + \sum_{i\in Z_2,\, j\in I_i} x_{i,j}\varsigma_{i,j}. \tag{4.64}$$

Take any central element

$$\kappa \in \mathcal{A}_0^- + \mathcal{A}_0^{(\iota+1)} \tag{4.65}$$

of $\mathcal{A}$. We define

$$\{f, g\}_1 = \langle (L, [\varsigma_{(f)}, (\kappa\varsigma_{(g)})_+] - [(\kappa\varsigma_{(f)})_-, \varsigma_{(g)}]\rangle \qquad \text{for } f, g\in\mathcal{P}. \tag{4.66}$$

Moreover, we define

$$\{f, g\}_2 = \langle (L\varsigma_{(f)})_-L - L(\varsigma_{(f)}L)_-, \varsigma_{(g)}\rangle \qquad \text{for } f, g\in\mathcal{P}. \tag{4.67}$$

The following is the main theorem in this section.

Theorem 4.3. *The brackets* $\{\cdot,\cdot\}_1$ *in (4.66) and* $\{\cdot,\cdot\}_2$ *in (4.67) forms a Poisson superpair on the algebra P.*

Proof. Let $\epsilon \in \mathbb{F}$ be any fixed constant. Set

$$\hat{L} = L + \epsilon\kappa. \tag{4.68}$$

Then we have

$$\partial_u(\hat{L}) = u \qquad \text{for } u \in \mathcal{G} \tag{4.69}$$

by (4.32) and (4.34). We define a map $H : \Omega \to \mathcal{G}$ by

$$H = (\hat{L}u)_-\hat{L} - \hat{L}(u\hat{L})_- \qquad \text{for } u \in \mathcal{G}. \tag{4.70}$$

For $\varsigma_\eta \in \Omega$, we have

$$H(\varsigma_\eta) = \sum_{i\in\mathbb{Z}_2,\ j\in I_i} \eta_{i,j}((\hat{L}\varsigma_{i,j})_-\hat{L} - \hat{L}(\varsigma_{i,j}\hat{L})_-). \tag{4.71}$$

Hence H is of the form (4.45) by (4.63), (4.64) and the fact that $\kappa \in \mathcal{A}_0$. Moreover, it is straightforward to verify that

$$H(u) = (Lu)_-L - L(uL)_+ + \epsilon(\kappa[L, u]_- + [(\kappa u)_-, L]) \qquad \text{for } u \in \mathcal{G}. \tag{4.72}$$

Thus by (2.32)–(2.34), (4.49), (4.58), (4.66) and (4.67), we have

$$\{\cdot, \cdot\}_H = \epsilon\{\cdot, \cdot\}_1 + \{\cdot, \cdot\}_2. \tag{4.73}$$

Therefore, we only need to prove that $\{\cdot,\cdot\}_H$ is a Poisson superbracket on $\mathcal{P}$. Lemma 4.1 tells us that it is enough to prove (4.50) and (4.51).

For $u \in \Omega_{i_1}$ and $v \in \Omega_{i_2}$, we have

$$\begin{aligned}
\langle H(v), u\rangle &= \langle(\hat{L}v)_-\hat{L} - \hat{L}(v\hat{L})_-, u\rangle \\
&= \langle[\hat{L}v - (\hat{L}v)_+]\hat{L} - \hat{L}[v\hat{L} - (v\hat{L})_+], u\rangle \\
&= -\langle(\hat{L}v)_+\hat{L} - \hat{L}(v\hat{L})_+, u\rangle \\
&= -\langle(\hat{L}v)_+\hat{L}, u\rangle + \langle\hat{L}(v\hat{L})_+, u\rangle \\
&= -\langle(\hat{L}v)_+, \hat{L}u\rangle + \langle(v\hat{L})_+, u\hat{L}\rangle \\
&= -\langle\hat{L}v, (\hat{L}u)_-\rangle + \langle v\hat{L}, (u\hat{L})_-\rangle \\
&= -(-1)^{i_1i_2}\langle(\hat{L}u)_-, \hat{L}v\rangle + \langle v, \hat{L}(u\hat{L})_-\rangle \\
&= -(-1)^{i_1i_2}(\langle(\hat{L}u)_-\hat{L}, v\rangle - \langle\hat{L}(u\hat{L})_-, v\rangle) \\
&= -(-1)^{i_1i_2}\langle(\hat{L}u)_-\hat{L} - \hat{L}(u\hat{L})_-, v\rangle \\
&= -(-1)^{i_1i_2}\langle H(u), v\rangle
\end{aligned} \tag{4.74}$$

by (2.33), (2.34), (4.58) and the supersymmetry and associativity of $\langle\cdot,\cdot\rangle$ on $\overline{\mathcal{G}}$. Thus (4.50) holds.

Let $u \in \Omega_{i_1},\ v \in \Omega_{i_2}$ and $w \in \Omega_{i_3}$. We have

$$\begin{aligned}
&\langle\partial_{H(u)}(H)(v), w\rangle \\
&\quad= \langle(H(u)v)_-\hat{L} + (-1)^{i_1i_2}(\hat{L}v)_-H(u) - H(u)(v\hat{L})_- \\
&\qquad - (-1)^{i_1i_2}\hat{L}(vH(u))_-, w\rangle \\
&\quad= \langle[((\hat{L}u)_-\hat{L} - \hat{L}(u\hat{L})_-)v]_-\hat{L} + (-1)^{i_1i_2}(\hat{L}v)_-((\hat{L}u)_-\hat{L} - \hat{L}(u\hat{L})_-) \\
&\qquad - ((\hat{L}u)_-\hat{L} - \hat{L}(u\hat{L})_-)(v\hat{L})_- - (-1)^{i_1i_2}\hat{L}[v((\hat{L}u)_-\hat{L} - \hat{L}(u\hat{L})_-)]_-, w\rangle \\
&\quad= \langle(((\hat{L}u)_-\hat{L}v)_-\hat{L}, w\rangle - \langle(\hat{L}(u\hat{L})_-v)_-\hat{L}, w\rangle + (-1)^{i_1i_2}\langle(\hat{L}v)_-(\hat{L}u)_-\hat{L}, w\rangle \\
&\qquad - (-1)^{i_1i_2}\langle(\hat{L}v)_-\hat{L}(u\hat{L})_-, w\rangle - \langle(\hat{L}u)_-\hat{L}(v\hat{L})_-, w\rangle + \langle\hat{L}(u\hat{L})_-(v\hat{L})_-, w\rangle \\
&\qquad - (-1)^{i_1i_2}\langle\hat{L}(v(\hat{L}u)_-\hat{L})_-, w\rangle + (-1)^{i_1i_2}\langle\hat{L}(v\hat{L}(u\hat{L})_-)_-, w\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle ((\hat{L}u)_-\hat{L}v)_-, \hat{L}w\rangle - \langle (\hat{L}(u\hat{L})_-v)_-, \hat{L}w\rangle + (-1)^{i_1i_2}\langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle\\
&\quad - (-1)^{i_1i_2}\langle (\hat{L}v)_-\hat{L}(u\hat{L})_-, w\rangle - \langle (\hat{L}u)_-\hat{L}(v\hat{L})_-, w\rangle + \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle\\
&\quad - (-1)^{i_1i_2}\langle (v(\hat{L}u)_-\hat{L})_-, w\hat{L}\rangle + (-1)^{i_1i_2}\langle (v\hat{L}(u\hat{L})_-)_-, w\hat{L}\rangle\\
&= \langle (\hat{L}u)_-\hat{L}v, (\hat{L}w)_+\rangle - \langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+\rangle + (-1)^{i_1i_2}\langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle\\
&\quad - (-1)^{i_1i_2}\langle (\hat{L}v)_-\hat{L}(u\hat{L})_-, w\rangle - \langle (\hat{L}u)_-\hat{L}(v\hat{L})_-, w\rangle + \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle\\
&\quad - (-1)^{i_1i_2}\langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_+\rangle + (-1)^{i_1i_2}\langle v\hat{L}(u\hat{L})_-, (w\hat{L})_+\rangle\\
&= (-1)^{i_1(i_2+i_3)}\langle \hat{L}v, (\hat{L}w)_+(\hat{L}u)_-\rangle + (-1)^{i_1i_2}\langle v\hat{L}, (u\hat{L})_-(w\hat{L})_+\rangle\\
&\quad - [\langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+\rangle + \langle (\hat{L}u)_-\hat{L}(v\hat{L})_-, w\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle]\\
&\quad - (-1)^{i_1i_2}[\langle (\hat{L}v)_-\hat{L}(u\hat{L})_-, w\rangle + \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_+\rangle - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle]
\end{aligned}
\tag{4.75}
$$

by (2.33), (2.34), (4.58) and the supersymmetry and associativity of $\langle\cdot,\cdot\rangle$ on $\overline{\mathcal{G}}$. Moreover,

$$
\begin{aligned}
&\langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+\rangle + \langle (\hat{L}u)_-\hat{L}(v\hat{L})_-, w\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}\\
&= \langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+\rangle + (-1)^{(i_1+i_2)i_3}\langle (\hat{L}w)_-\hat{L}(u\hat{L})_-, v\rangle\\
&\quad - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}\\
&= \langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+\rangle + \langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_-\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}\\
&= \langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+ + (\hat{L}w)_-\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}\\
&= \langle \hat{L}(u\hat{L})_-v, \hat{L}w\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}\\
&= \langle (u\hat{L})_-v\hat{L}, w\hat{L}\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}\\
&= \langle (u\hat{L})_-(v\hat{L})_+, w\hat{L}\rangle + \text{c.p.}\\
&= \langle u\hat{L}v\hat{L}, w\hat{L}\rangle
\end{aligned}
\tag{4.76}
$$

and

$$
\begin{aligned}
&\langle (\hat{L}v)_-\hat{L}(u\hat{L})_-, w\rangle + \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_+\rangle - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= (-1)^{i_2(i_1+i_3)}\langle (\hat{L}u)_-\hat{L}(w\hat{L})_-, v\rangle + \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_+\rangle\\
&\quad - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_-\rangle + \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_+\rangle - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_- + (w\hat{L})_+\rangle - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= \langle v(\hat{L}u)_-\hat{L}, w\hat{L}\rangle - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= \langle \hat{L}v(\hat{L}u)_-, \hat{L}w\rangle - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= \langle (\hat{L}v)_+(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}\\
&= \langle \hat{L}v\hat{L}u, \hat{L}w\rangle
\end{aligned}
\tag{4.77}
$$

by (4.55)–(4.58) and the supersymmetry and associativity of $\langle\cdot,\cdot\rangle$ on $\overline{\mathcal{G}}$. Thus

$$
\begin{aligned}
&\langle \partial_{H(u)}(H)(v), w\rangle + \text{c.p.}\\
&= [(-1)^{i_1(i_2+i_3)}\langle \hat{L}v, (\hat{L}w)_+(\hat{L}u)_-\rangle + \text{c.p.}]\\
&\quad + [(-1)^{i_1i_2}\langle v\hat{L}, (u\hat{L})_-(w\hat{L})_+\rangle + \text{c.p.}]\\
&\quad - [\langle \hat{L}(u\hat{L})_-v, (\hat{L}w)_+\rangle + \langle (\hat{L}u)_-\hat{L}(v\hat{L})_-, w\rangle - \langle (u\hat{L})_-(v\hat{L})_-, w\hat{L}\rangle + \text{c.p.}]\\
&\quad - (-1)^{i_1i_2}[\langle (\hat{L}v)_-\hat{L}(u\hat{L})_-, w\rangle + \langle v(\hat{L}u)_-\hat{L}, (w\hat{L})_+\rangle\\
&\quad - \langle (\hat{L}v)_-(\hat{L}u)_-, \hat{L}w\rangle + \text{c.p.}]\\
&= \langle \hat{L}u, \hat{L}v\hat{L}w\rangle + (-1)^{i_1i_2}\langle v\hat{L}, u\hat{L}w\hat{L}\rangle\\
&\quad - \langle u\hat{L}v\hat{L}, w\hat{L}\rangle - (-1)^{i_1i_2}\langle \hat{L}v\hat{L}u, \hat{L}w\rangle = 0
\end{aligned}
\tag{4.78}
$$

by (4.56)–(4.58), the supersymmetry and associativity of $\langle \cdot, \cdot \rangle$ on $\overline{\mathcal{G}}$ and (4.75)–(4.77). Therefore, (4.51) holds. □

5. Hamiltonian superpairs in variational calculus

In this section, we shall construct certain Hamiltonian superpairs in the formal variational calculus over a finite-dimensional $\mathbb{Z}_2$-graded associative algebras with a supersymmetric nondegenerate associative bilinear form. We assume that $\mathbb{F}$ is the field of real numbers or the field of complex numbers.

Let $\mathcal{A}$ be a finite-dimensional $\mathbb{Z}_2$-graded associative algebra with a supersymmetric nondegenerate associative bilinear form $\langle \cdot, \cdot \rangle$. Denote

$$\dim \mathcal{A}_i = k_i \qquad \text{for } i \in \mathbb{Z}_2. \tag{5.1}$$

Fix a nonnegative intger ι. Set

$$I_i = \overline{1, k_i} \times (-\mathbb{N} \bigcup \overline{1, \iota}) \qquad \text{for } i \in \mathbb{Z}_2. \tag{5.2}$$

For $i \in \mathbb{Z}_2$, let $\{u_{i,j} \mid j \in I_i\}$ be the C^∞-functions in real variable x, taking values in $\mathcal{E}_i$ (cf. (4.2)). Set

$$\partial = \frac{\mathrm{d}}{\mathrm{d}x} \qquad u_{i,j}^{(m)} = \frac{\mathrm{d}^m u_{i,j}}{\mathrm{d}x^m} \qquad \text{for } m \in \mathbb{N},\ i \in \mathbb{Z}_2,\ j \in I_i. \tag{5.3}$$

Denote by $\mathcal{P}$ the algebra supersymmetric polynomials in $\{u_{i,j}^{(m)} \mid i \in \mathbb{Z}_2,\ j \in I_i\}$. Then $\mathcal{P}$ has the $\mathbb{Z}_2$-grading

$$\mathcal{P}_i = \mathrm{Span}\left\{ u_{i_1,j_1}^{(m_1)} \cdots u_{i_\ell,j_\ell}^{m_\ell} | \ell, m_p \in \mathbb{N}, i_p \in \mathbb{Z}_2, j_p \in I_{i_p}, \sum_{r=1}^{\ell} i_r \equiv i \right\} \tag{5.4}$$

for $i \in \mathbb{Z}_2$, and (4.14) holds.

Take a basis $\{\varsigma_{i,l} \mid l \in \overline{1, k_i}\}$ of $\mathcal{A}_i$ for $i \in \mathbb{Z}_2$. Let $\hat{\mathcal{A}}$ be the free $\mathcal{P}$-module generated by $\{\varsigma_{i,l} \mid l \in \overline{1, k_i}\}$ with the $\mathbb{Z}_2$-grading

$$\hat{\mathcal{A}}_i = \sum_{j \in \overline{1,k_0}} \mathcal{P}_i \varsigma_{0,j} + \sum_{j \in \overline{1,k_1}} \mathcal{P}_{1+i} \varsigma_{1,j} \qquad \text{for } i \in \mathbb{Z}_2. \tag{5.5}$$

Moreover, we define the multiplication on $\hat{\mathcal{A}}$ by

$$(f \varsigma_{i_1,j_1})(g \varsigma_{i_2,j_2}) = (-1)^{i_1 p} f g \varsigma_{i_1,j_1} \varsigma_{i_2,j_2} \qquad \text{for } i_q \in \mathbb{Z}_2,\ j_q \in \overline{1, k_{i_q}},\ g \in \mathcal{P}_p \tag{5.6}$$

and the bilinear form on $\hat{\mathcal{A}}$ by

$$\langle f \varsigma_{i_1,j_1}, g \varsigma_{i_2,j_2} \rangle = (-1)^{i_1 p} f g \langle \varsigma_{i_1,j_1}, \varsigma_{i_2,j_2} \rangle \qquad \text{for } i_q \in \mathbb{Z}_2,\ j_q \in \overline{1, k_{i_q}},\ g \in \mathcal{P}_p \tag{5.7}$$

Then $\hat{\mathcal{A}}$ forms a $\mathbb{Z}_2$-graded associative algebra with a supersymmetric nondegenerate associative bilinear form $\langle \cdot, \cdot \rangle$. The algebra $\hat{\mathcal{A}}$ is an extension of $\mathcal{A}$ with extended supersymmetric bilinear form $\langle \cdot, \cdot \rangle$.

We view

$$\partial = \sum_{i \in \mathbb{Z}_2,\ j \in \overline{1,k_i},\ m \in \mathbb{N}} u_{i,l}^{(m+1)} \partial_{u_{i,j}^{(m)}} \tag{5.8}$$

as a derivation of $\mathcal{P}$. Moreover, we extend ∂ to a derivation of $\hat{\mathcal{A}}$ by

$$\partial(f \varsigma_{i,j}) = \partial(f) \varsigma_{i,j} \qquad \text{for } i \in \mathbb{Z}_2,\ j \in \overline{1, k_i}. \tag{5.9}$$

Furthermore, we define the algebra of pseudo-differential operators

$$\mathcal{D}=\left\{\sum_{l=-\infty}^{n}\phi_l\partial^l \mid n\in\mathbb{Z},\ \phi_l\in\hat{\mathcal{A}}\right\} \tag{5.10}$$

with multiplication determined by

$$(\phi\partial^m)(\psi\partial^n)=\sum_{p=0}^{\infty}\binom{m}{p}\phi\psi^{(p)}\partial^{m+n-p} \qquad \text{for } \phi,\psi\in\hat{\mathcal{A}},\quad m,n\in\mathbb{Z} \tag{5.11}$$

where

$$\psi^{(p)}=\partial^p(\psi). \tag{5.12}$$

Then $\mathcal{D}$ forms a $\mathbb{Z}_2$-graded associative algebra with the grading

$$\mathcal{D}_i=\left\{\sum_{l=-\infty}^{n}\phi_l\partial^l \mid n\in\mathbb{Z},\ \phi_l\in\hat{\mathcal{A}}_i\right\} \qquad \text{for } i\in\mathbb{Z}_2. \tag{5.13}$$

Define the space

$$\mathcal{G}=\left\{\sum_{l=-\infty}^{\iota}\phi_l\partial^l \mid \phi_l\in\hat{\mathcal{A}}\right\}. \tag{5.14}$$

Then $\mathcal{G}$ is a $\mathbb{Z}_2$-graded subspace of $\mathcal{D}$ with the grading

$$\mathcal{G}_0=\mathcal{G}\bigcap\mathcal{D}_0, \qquad \mathcal{G}_1=\mathcal{G}\bigcap\mathcal{D}_1. \tag{5.15}$$

For

$$v=\sum_{i\in\mathbb{Z}_2,(j_1,j_2)\in I_i} f_{i,(j_1,j_2)}\varsigma_{i,j_1}\partial^{j_2}\in\mathcal{G} \qquad \text{with } f_{i,(j_1,j_2)}\in\mathcal{P} \tag{5.16}$$

we define the derivation of $\mathcal{P}$:

$$\partial_v=\sum_{i\in\mathbb{Z}_2,\ j\in I_i,\ m\in\mathbb{N}} f_{i,j}^{(m)}\partial_{u_{i,j}^{(m)}} \tag{5.17}$$

(cf. (5.2)). It can be verified that as derivations on $\mathcal{P}$,

$$[\partial,\partial_v]=0 \qquad \text{for } v\in\mathcal{G}. \tag{5.18}$$

Moreover, we set

$$\partial_v(w)=\sum_{i\in\mathbb{Z}_2,(j_1,j_2)\in I_i}\partial_v(g_{i,(j_1,j_2)})\varsigma_{i,j_1}\partial^{j_2} \tag{5.19}$$

for $v,w=\sum_{i\in\mathbb{Z}_2,(j_1,j_2)\in I_i} g_{i,(j_1,j_2)}\varsigma_{i,j_1}\partial^{j_2}\in\mathcal{G}$. Furthermore, we can define a Lie superbracket on $\mathcal{G}$ by

$$[v,w]_0=\partial_v(w)-(-1)^{i_1i_2}\partial_w(v) \qquad \text{for } v\in\mathcal{G}_{i_1},\ w\in\mathcal{G}_2. \tag{5.20}$$

Set

$$\tilde{\mathcal{P}}=\mathcal{P}/\partial(\mathcal{P}) \tag{5.21}$$

and denote

$$\tilde{f}=f+\partial(\mathcal{P}) \qquad \text{for } f\in\mathcal{P}. \tag{5.22}$$

We define an action of $\mathcal{G}$ on $\tilde{\mathcal{P}}$ by

$$v(\tilde{f})=(\partial_v(f))^{\sim} \qquad \text{for } f\in\mathcal{P}. \tag{5.23}$$

Then $\tilde{\mathcal{P}}$ becomes a $\mathcal{G}$-module. Moreover, we define a bilinear map $\langle\cdot,\cdot\rangle : \mathcal{D}\times\mathcal{D}\to\tilde{\mathcal{P}}$ by

$$\langle v, w\rangle = \sum_{j\in\mathbb{Z}} (\langle\phi_j, \psi_{-j-1}\rangle)^{\sim} \qquad \text{for} \quad v = \sum_{m\in\mathbb{Z}} \phi_m \partial^m, \quad w = \sum_{m\in\mathbb{Z}} \psi_m \partial^m \in \mathcal{D} \tag{5.24}$$

(cf. (5.7)), where the sum is finite by (5.10). Then $\langle\cdot,\cdot\rangle$ forms a supersymmetric associative bilinear map (cf. (2.32)) by lemma 2.6 in [10].

For $v = \sum_{j\in\mathbb{Z}} \phi_j \partial^j \in \mathcal{D}$, we define

$$v_+ = \sum_{j=0}^{\infty} \phi_j \partial^j \qquad v_- = \sum_{j=1}^{\infty} \phi_{-j}\partial^{-j}. \tag{5.25}$$

Set

$$\mathcal{D}_\pm = \{v_\pm \mid v \in \mathcal{D}\}. \tag{5.26}$$

Then $\mathcal{D}^\pm$ are isotropic $\mathbb{Z}_2$-graded subalgebra of $\mathcal{D}$ with respect to the bilinear map $\langle\cdot,\cdot\rangle$. Define

$$\Omega = \sum_{m=-\iota-1}^{\infty} \partial^m \hat{\mathcal{A}}. \tag{5.27}$$

Then Ω is a $\mathbb{Z}_2$-graded subspace of $\mathcal{D}$ by (5.11) with the grading

$$\Omega_0 = \Omega \bigcap \mathcal{D}_0 \qquad \Omega_1 = \Omega \bigcap \mathcal{D}_1. \tag{5.28}$$

Identify Ω with a subspace of one-forms (taking values in $\tilde{\mathcal{P}}$ by

$$\varpi(v) = \langle v, \varpi\rangle \qquad \text{for } v \in \mathcal{G},\ \varpi \in \Omega. \tag{5.29}$$

We define *variational operators* on $\mathcal{P}$ by

$$\delta_{(i,j)} = \sum_{m=0}^{\infty} (-\partial)^m \circ \partial_{u_{i,j}^{(m)}} \qquad \text{for} \quad i \in \mathbb{Z}_2,\ j \in I_i \tag{5.30}$$

where $\circ$ is the composition of operators on $\mathcal{P}$. It can be verified that

$$\delta_{(i,j)}(\partial(\mathcal{P})) = \{0\} \qquad \text{for } i \in \mathbb{Z}_2,\ j \in I_i. \tag{5.31}$$

Moreover, we define a linear map $\chi : \mathcal{P} \to \Omega$ by

$$\chi_f = \sum_{i\in\mathbb{Z},\ j=(j_1,j_2)\in I_i} (-1)^{i(1+p)} \partial^{-j_2-1} \delta_{(i,j)}(f) \varsigma_{i,j_1} \qquad \text{for} \quad f \in \mathcal{P}_p. \tag{5.32}$$

Then we can verify

$$df(v) = \langle v, \chi_f\rangle \qquad \text{for} \quad f \in \mathcal{P},\ v \in \mathcal{G}. \tag{5.33}$$

Hence

$$df = \chi_f \in \Omega \qquad \text{for} \quad f \in \mathcal{P}. \tag{5.34}$$

The Lie superbracket on $\mathcal{D}$ is defined by

$$[v_1, v_2] = v_1 v_2 - (-1)^{i_1 i_2} v_2 v_1 \qquad \text{for} \quad v_1 \in \mathcal{P}_{i_1},\ v_2 \in \mathcal{P}_{i_2}. \tag{5.35}$$

Take

$$L_0 = \sum_{m=-\infty}^{\iota+1} \sigma_m \partial^m \qquad \text{with} \quad \sigma_m \in \mathcal{A}_0 \tag{5.36}$$

and

$$\kappa \in \mathcal{A}_0 \bigcap (\text{Centre } \mathcal{A}). \tag{5.37}$$

Set

$$L = L_0 + \sum_{i\in\mathbb{Z}_2,\ j=(j_1,j_2)\in I_i} u_{i,j}\varsigma_{i,j_1}\partial^{j_2} \tag{5.38}$$

(cf. (5.2)). We define the linear maps $H_1, H_2 : \Omega \to \mathcal{G}$ by

$$\begin{aligned} H_1(\varpi) &= \kappa[L, \varpi]_- + [(\kappa\varpi)_-, L] \\ H_2(\varpi) &= (L\varpi)_- L - L(\varpi L)_- \qquad \text{for } \varpi \in \Omega. \end{aligned} \tag{5.39}$$

By similar arguments as those in the proof of theorem 4.3, we have:

Theorem 5.1. *The pair (H_1, H_2) forms a Hamiltonian superpair.*

The examples of finite-dimensional $\mathbb{Z}_2$-graded associative algebras with a supersymmetric nondegenerate associative bilinear form were given in example 3.2. In fact, they contain an identity element (the identity matrix).

Remark 5.2. Suppose that $\mathcal{A}$ contains an identity element. Take the special case $L_0 = \partial^{\iota+1}$. Then

$$L = \partial^{\iota+1} + \sum_{i\in\mathbb{Z}_2,\ j=(j_1,j_2)\in I_i} u_{i,j}\varsigma_{i,j_1}\partial^{j_2}. \tag{5.40}$$

By an algebraic manipulation, we can find

$$L^{1/(\iota+1)} = \partial + \sum_{m=0}^{\infty} f_m\partial^{-m} \qquad \text{with } f_m \in \hat{\mathcal{A}} \tag{5.41}$$

such that

$$(L^{1/(\iota+1)})^{\iota+1} = L. \tag{5.42}$$

Since $\mathcal{A}$ has an identity element $1_{\mathcal{A}}$, the map

$$\mathrm{Tr}\,(a) = \langle 1_{\mathcal{A}}, a\rangle \qquad \text{for } a \in \mathcal{A} \tag{5.43}$$

is a supersymmetric trace map, that is,

$$\mathrm{Tr}\,(ab) = (-1)^{i_1 i_2}\mathrm{Tr}\,(ba) \qquad \text{for } a \in \mathcal{A}_{i_1},\ b \in \mathcal{A}_{i_2}. \tag{5.44}$$

We extend the trace map to $\mathcal{D}$ by

$$\mathrm{Tr}\left(\sum_{i\in\mathbb{Z}_i,\ l\in\overline{1,k_i},\ m\in\mathbb{Z}} f_{i,l,m}\varsigma_{i,l}\partial^m\right) = \sum_{i\in\mathbb{Z}_i,\ l\in\overline{1,k_i}} f_{i,l,-1}\mathrm{Tr}\,(\varsigma_{i,l}). \tag{5.45}$$

Then the above map is a supersymmetric trace map of the associative algebra $\mathcal{D}$ by lemma 2.6 in [10].

Assume that $\{u_{i,j} \mid i \in \mathbb{Z}_2,\ j \in I_i\}$ are also C^1-functions of another variable t and periodic in x. Take a positive integer m. Then the system

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [L, (L^{m/(\iota+1)})_+] \tag{5.46}$$

has infinitely many conservation laws:

$$\mathrm{Tr}\,(L^{n/(\iota+1)}) \qquad \text{for } n \in \mathbb{N}+1. \tag{5.47}$$

Suppose that $\{u_{i,j} \mid i \in \mathbb{Z}_2,\ j \in I_i\}$ are C^1-functions of variable $\{t_1, t_2, t_3, \ldots\}$ and periodic in x. Set

$$B_m = (L^{m/(\iota+1)})_+ \qquad \text{for } m \in \mathbb{N}+1. \tag{5.48}$$

Assume that

$$\frac{\mathrm{d}L}{\mathrm{d}t_m} = [L, B_m] \qquad \text{for } m \in \mathbb{N}+1. \tag{5.49}$$

Then

$$\frac{\mathrm{d}B_m}{\mathrm{d}t_n} - \frac{dB_n}{dt_m} = [B_m, B_n] \qquad \text{for } m, n \in \mathbb{N}+1. \tag{5.50}$$

The above equations are called the *equations of zero curvature*. We refer to [7, 8, 10] for more details.

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